"Knowing mathematics is like wearing a pair of X-ray specs that reveal hidden structures underneath the messy and chaotic surface of the world."

How Not to Be Wrong: The Power of Mathematical Thinking
by Jordan Ellenberg

ASSESSING SECONDARY TEACHERS' ALGEBRAIC HABITS OF MIND

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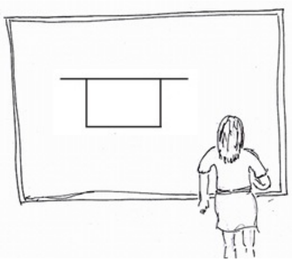
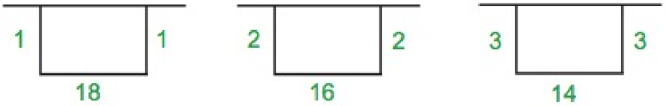
Generalizing from repeated reasoning



The class is considering the following problem:

> Barry has 20 m of wire mesh. He would like to enclose as large as possible a rectangular chicken coop using a barn wall as one of the sides. What values for the length and width give him the largest area?

GENERALIZING FROM REPEATED REASONING

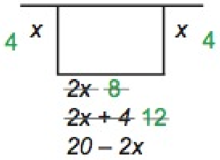


T: Just kind of look at it. If the length is just 1, then what does the width have to be?

S: Is it 9? Oh, it has to be 18. I thought there were four sides.

T: So when the length is 1, the width has to be 18, since they have to add up to 20. What if the length is 2? What if it's 3?

GENERALIZING FROM REPEATED REASONING



Students begin to recognize a linear pattern in widths. The teacher asks for the width if the length is x. Students first say $x - 2$, then $2x$. The teacher suggests they check $2x$ by substituting 4 for x.

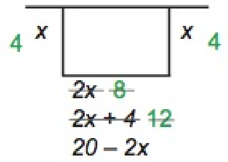
T: So if $x = 4$, the width will be 8. Is that true?

S: No, you have to plus something.

T: How do you figure it out? What is it supposed to be if it's 4?

S: 12.

GENERALIZING FROM REPEATED REASONING

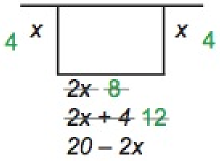


Students then suggest $2x + 4$ (to get 12), but they realize their error after substituting 5 for x.

T: So we know it's actually going down by 2. [Student name], why did you pick $2x$?

S: Because there are two sides for the x. Oh, you minus 20 to give you the width.

GENERALIZING FROM REPEATED REASONING



Students begin to articulate that to find the width, they had been multiplying the length by 2 and then subtracting that from 20, which is the amount of wire that was initially given.

Why habits of mind?

- This teacher has participated in many of our PD programs with explicit emphasis on *mathematical habits of mind* (MHoM).
- Current standards are aligned with this way of thinking. Teachers need MHoM because they are expected to provide opportunities for their students to develop those habits.
- Furthermore, many other teachers have reported to us that developing their MHoM have strongly impacted their teaching.
- We've seen that MHoM are *habits* that teachers can acquire, rather than some static you-have-it-or-you-don't way of thinking.

Initial motivation for research

- We recognized the need to investigate how teachers use MHoM in their teaching, and how those habits can change over time.
- The instruments to measure these habits did not yet exist.
- Thus, we began a research study centered on the question:

  What mathematical habits of mind do secondary teachers use, how do they use them, and how can we measure them?

RESEARCH INSTRUMENTS

To investigate our research question, we've been developing:

- Detailed definition of MHoM, based on existing literature, our own experiences as mathematicians, and classroom observations.
- An observation framework for understanding the nature of teachers' use of MHoM in their classroom work.
- A paper and pencil (P&P) assessment that measures how teachers engage MHoM when doing mathematics for themselves.

Note: We have developed all three components together.

Important remarks

- Our original intent was to assess our *own* PD programs and learn how to strengthen their impact.
- We are creating instruments for research and development, *not* for teacher evaluation. The instruments are intended to help researchers, district leaders, and PD developers better understand and meet the mathematical needs of secondary teachers.
- Our research is centered on understanding the nature of MHoM and the roles these habits play in teaching. We recognize that MHoM constitute just one aspect of a broad spectrum of knowledge and skills that teachers bring to their profession.

MHoM definition

We define mathematical habits of mind (MHoM) to be:

> *the specialized ways of approaching mathematical problems and thinking about mathematical concepts that resemble the ways employed by mathematicians.*

Focus on MHoM

Our current focus is on three categories of MHoM:

- **EXPR.** Engaging with one's experiences
- **STRC.** Making use of structure to solve problems
- **LANG.** Using mathematical language precisely

Remark: Focusing on three habits has allowed us to create a P&P assessment that is not too burdensome to use and has focused our observation work. Eventually, we will investigate other habits, too.

Connection to CCSSM

Our three mathematical habits are closely related to the following Common Core Standards for Mathematical Practice:

- **MP2.** Reason abstractly and quantitatively
- **MP6.** Attend to precision
- **MP7.** Look for and make use of structure
- **MP8.** Look for and express regularity in repeated reasoning

Goals of observation work

- To collect examples of how secondary teachers use MHoM in the classroom, with an eye towards describing more generally what we mean by "classroom use of MHoM."
- To study the bridge factors between teachers' own MHoM (as measured by the P&P assessment) and their use of these habits in the classroom.

For you to do

We will show a video clip from an Algebra 1 classroom, which highlights a teacher providing opportunities for his students to develop / use MHoM.

Question to consider:

- What evidence do you see of the teacher offering students opportunities to develop and/or use MHoM?

What's happening in the video

In this video clip, students are working on the following problem:

> Think of two numbers and write them down as outputs $f(0)$ and $f(1)$. To find the next output, $f(2)$, subtract the first number from the second number. Continue this pattern to find additional outputs and make a table of the outputs.
>
> - What patterns do you notice in the output?
> - What recursive rule describes the function?
> - Repeat the process with different starting numbers. Are some starting numbers special?

QUESTION #1

What evidence do you see of the teacher offering students opportunities to develop and/or use MHoM?

Further questions

- How would you capture that evidence on paper?
- Was there something in this episode that made you say, "Yes, that's what exhibiting MHoM looks like"? If so, what?
- Could you give a general description of what you saw that might help others see what you're seeing?
- Could you imagine seeing the same mathematical habit being used in a different classroom? If so, how?

Note: We're not expecting resolutions to these questions.

P&P assessment: Overview

- We are developing a P&P assessment that measures how teachers engage MHoM when doing mathematics for themselves.
- The assessment has been field-tested with over 500 teachers. Field-tests are ongoing.
- Initial validity and reliability testing yielded promising results. More testing is being planned.
- Again, this is a tool for research, *not* for teacher evaluation.

P&P ASSESSMENT: KEY FEATURES

- Assessment measures how secondary teachers use mathematical habits of mind when doing mathematics.
- Items are accessible: most secondary teachers can solve them, or at least begin to solve them.
- Coding focuses on the *approach*, not on "the correct solution."
- Assessment items are drawn from multiple sources, including our classroom observation work.

Maximum Value

Sample Item:

Find the maximum value of the function $f(x) = 11 - (3x - 4)^2$.

- Though most teachers obtained the same (correct) answer, there were vast variations in their approaches.
- These various approaches came in "clumps," as our advisors (assessment experts) and research literature had told us to expect.
- Using these responses, we developed a rubric that allows us to code how each teacher solved the problem.

SAMPLE CODE: SQUR

SQUR: Since $(3x-4)^2$ represents the ***square*** of some number, it is always ≥ 0. Thus in the function $f(x) = 11 - (3x-4)^2$, we are always subtracting a non-negative number from 11. To maximize $f(x)$, we need $(3x-4)^2 = 0$ so that the maximum value is 11.

Sample solution:

$f(x) = 11-(3x-4)^2$. Anything squared is ≥ 0.

Therefore, 11 − (stuff squared) must be ≤ 11. So 11 is the max.

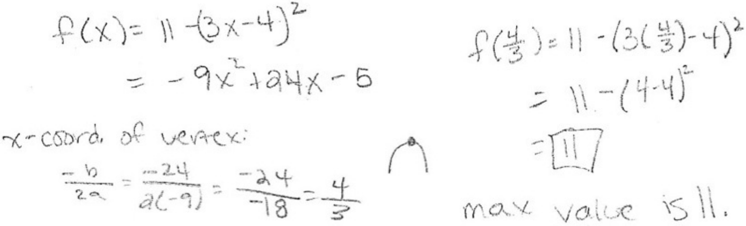
QUICK MATHEMATICAL NOTE

The reasoning described in **SQUR** depends on the fact that x can be chosen so that $(3x-4)^2 = 0$. In many cases, we had no way of knowing whether the teachers actually noticed this detail.

SAMPLE CODE: SYMM

SYMM: Expands $f(x)$ into $f(x) = -9x^2 + 24x - 5$. Finds the axis of ***symmetry*** using the formula $x = -b/(2a) = 4/3$. Evaluates $f(4/3) = 11$ to obtain the maximum value.

Sample solution:



Maximum Value item

- How clear is the description of each code? Can we revise them to provide more clarity? If so, how?
- Do the ways in which you think about this item match the habit that we claim it measures, namely **STRC**?
- What should we do with responses like 7_10 and 7_18 where they set $3x - 4 = 0$ without any explanation of why? They don't seem deserving of **SQUR**. Should it be a separate code? Or **NEVD**?

Three Products item

- How clear is the description of each code? Can we revise them to provide more clarity? If so, how?
- Do the ways in which you think about this item match the habit that we claim it measures, namely **EXPR**?
- For the code/approach **SMEX** (smaller examples):
  - What if examples are generated but are interpreted incorrectly?
  - Is it really okay if a conclusion is made from just one example?
- Is an area model the same as **PLMN**? Or assign a different code?
- What constitutes as an *evidence* of, say, **2VAR**? How much work needs to be shown to convey intent? (E.g., 7_04, 7_36.)

Sum of Squares item

- How clear is the description of each code? Can we revise them to provide more clarity? If so, how?
- Do the ways in which you think about this item match the habit that we claim it measures, namely **STRC**?
- **SOLV** vs. **REDC** could be a difficult distinction. What constitutes as an evidence of an attempt to solve for a and b? How much work needs to be shown to convey intent?

LEARN MORE OR PARTICIPATE

Want to learn more, use the assessment, or participate in the research?

mhomresearch.edc.org

THANK YOU

- Thank you for your participation and feedback!
- If you have further feedback and/or questions, email us at:
  - Sarah Sword (ssword@edc.org)
  - Ryota Matsuura (matsuura@stolaf.edu)