Journal of Mathematical Behavior 57 (2020) 100751

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Eliminating counterexamples: A case study intervention for improving adolescents' ability to critique direct arguments

David A. Yopp[a,*], Rob Ely[b], Anne E. Adams[c], Annelise W. Nielsen[b], Erin C. Corwine[d]

[a] Departments of Mathematics and Curriculum and Instruction, University of Idaho, 875 Perimeter Drive MS 1103, Moscow, ID 83844-1103, United States

[b] Department of Mathematics, University of Idaho, 875 Perimeter Drive MS 1103, Moscow, ID 83844-1103, United States

[c] Department of Curriculum and Instruction, University of Idaho, 875 Perimeter Drive MS 3082, Moscow, ID 83844-3082, United States

[d] Palouse Prairie Charter School, 500 S. Levick St., Moscow, ID 83843, United States

ARTICLE INFO

Keywords:
Reasoning
Proving
Contrapositive proving
Teaching intervention
Viable argument
Counterexamples

ABSTRACT

Students' difficulties with argumentation, proving, and the role of counterexamples in proving are well documented. Students in this study experienced an intervention for improving their argumentation and proving practices. The intervention included the *eliminating counterexamples* (ECE) framework as a means of constructing and critiquing viable arguments for a general claim. This framework involves constructing descriptions of all possible counterexamples to a conditional claim and determining whether or not a direct argument eliminates the possibility of counterexamples. This case study investigates U.S. eighth-grade (age 13) mathematics students' conceptions about the validity of a direct argument after the students received instruction on the ECE framework. We describe student activities in response to the intervention, and we identify students' conceptions that are inconsistent with canonical notions of mathematical proving and appear to be barriers to using the ECE framework.

1. Introduction

Yopp (2017) demonstrated that the *eliminating counterexamples* (ECE) framework for viable argumentation (e.g., proof/proving) for a general claim was used by a Grade 8 mathematics student (U.S. system, age 13) as she constructed and critiqued indirect arguments (e.g., contrapositive arguments, or proofs by contrapositive). This framework was grounded in the principle that in mathematics, quantified general claims are either true or false and that a general claim is true if and only if no counterexample to the general claim exists. In practice, the ECE framework for proof and proving entailed describing the class of counterexamples to a general claim and then either (a) showing the claim is false by finding an example that fits this description (a counterexample) or by theoretically demonstrating that a counterexample must exist, or (b) showing the claim is true by proving that no counterexample exists. Further nuances and elements of the ECE framework are detailed in Section 2, Background and Framing, and Section 3, Method, after more context and motivation for the approach have been provided.

The ECE framework was originally developed by the first author (Yopp, 2017) as a way to explain to Grade 8 students, who in general do not have access to formal logic, why indirect argument methods such as proof by contrapositive and proof by contradiction are viable arguments for the original claim. However, as Yopp (2017) notes, students who were taught the ECE framework also

* Corresponding author.
E-mail addresses: dyopp@uidaho.edu (D.A. Yopp), ely@uidaho.edu (R. Ely), aeadams@uidaho.edu (A.E. Adams), annelise@uidaho.edu (A.W. Nielsen), ecorwine@palouseprairieschool.org (E.C. Corwine).

https://doi.org/10.1016/j.jmathb.2019.100751
Received 15 October 2018; Received in revised form 2 November 2019; Accepted 7 November 2019
Available online 14 November 2019
0732-3123/

successfully applied it to evaluate direct arguments. These student activities provided insights into the broader possibilities of the framework.

The way the ECE framework supports direct arguments is less obvious than for indirect arguments but is arguably even more important, since direct arguments are more prevalent than indirect ones. The way the ECE approach can support reasoning in direct argumentation contexts warrants a rich exploration, one that until now has not been conducted.

In this article, we address that need for further research by examining how the ECE framework was used by a diverse group of Grade 8 mathematics students as they critiqued a direct argument. We study the conceptions students presented as they critiqued a direct argument for a general claim, after the students experienced a viable argumentation intervention that involved the ECE framework. In particular, we seek to examine the intermediate conceptions (ICs) students developed in response to an intervention for improving students' viable argumentation knowledge and practices in Grade 8 mathematics. This study investigates whether those conceptions supported or interfered with students' use of the ECE framework to validate a direct argument for a general claim. These research objectives are stated more precisely as research questions in Section 3, Method.

Throughout this article, we refer to *proof* and *viable argument/argumentation.* We use the term *proof* to refer to arguments for or against a mathematical statement that use logic and prior results. Grade 8 students may not have access to formal training in mathematical logic, so we use the term *viable argument* to refer to an argument that is proof-like but less than formal. For instance, a viable argument may express the same kind of reasoning as expressed in proofs but may use informal representations and may lack explicit acknowledgement of some prior results used to make inferences. From this perspective, all proofs are viable arguments, but some viable arguments may not be viewed by researchers and mathematicians as being complete proofs. Because argument refers to a product, the term *argumentation* is used to refer to any and all activities associated with constructing and critiquing an argument.

Using the term *viable argument* when discussing proof and proving is particularly important to us as researchers based in the United States. Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010) uses this term to describe practices K – 12 students will experience and express as they learn mathematics. These practices are akin to proof and proving practices, and we make the assumption that the recommended practices are intended to prepare students for making proofs.

The idea of the ECE framework for viable argumentation is that a description of possible counterexamples to a general claim can be developed and used to construct and critique an argument. When using the ECE framework to construct an argument for a general claim, the reasoner works with the description of possible counterexamples to either construct a counterexample, develop a non-constructive argument for the existence of a counterexample, or construct an argument for why a counterexample cannot exist. When using the ECE framework to critique an argument for a general claim, the arguer evaluates the argument against a standard of whether or not the argument demonstrates that all examples meeting the description of counterexamples are impossible.

Eliminating the possibility of counterexamples is not a stand-alone criterion for viable argumentation. Viable argumentation also involves a variety of other standards. These standards include norms for representing the objects in the domain of the claim, accepted methods (or modes) of argumentation, and acknowledgement of the prior results needed to make inferences. The ECE framework serves as an overarching definition of proof and viable argumentation for general claims that expresses a goal and purpose of viable argumentation and proof. Other tools and standards serve as means of accomplishing this goal of ECE and norms for communicating the argument. The following hypothetical example illustrates how notions and practices associated with the ECE framework might be used in a direct argument context.

Consider a hypothetical scenario where the general claim, "If you multiply two square numbers, you will get another square number," is presented to a Grade 8 student who does not know whether the claim is true or false. By *square number,* we mean numbers that are the square of some natural number. Using the ECE framework, the student constructs a description of possible counterexamples, "two numbers with two properties: both numbers are square numbers and the product is not a square." The student constructs several examples, wondering if she might find a counterexample or find a structural reason for why counterexamples cannot exist. Not finding a counterexample among the examples tested, the student notices some interesting structure. The product of 25 and 36, written as $5^2 6^2$, makes the meaning of the conditions, products of two square numbers, transparent. In this expression, the student sees a reason that the product is a square number. By applying an exponent property and the associative property, she transforms the expression into 30^2. The student realizes that this structure is shared by all possible examples of the conditions and that the algebraic transformation used in this example can be applied to all cases of products of two square numbers, making cases of "two square numbers" and "a product that is not a perfect square" impossible.

The above example illustrates how the ECE approach might be applied when constructing a direct argument (or proof). Now consider another student who critiques this argument. The student wonders if the argument is viable (or proof) and assesses whether or not counterexamples have been eliminated. She constructs a description of all possible counterexamples, examples of two numbers having two properties, "two square numbers" and "a product that is not a square number." She assesses whether or not the argument demonstrates that such examples are impossible by verifying that the representation, structure, and transformations apply to all cases of two square numbers. With these traits verified, the student concludes that the argument demonstrates examples with both properties of a counterexample cannot exist. Thus, she labels the argument as viable (or proof).

There are relatively few articles suggesting concrete ways to improve Grade 8 mathematics students' understanding of proof, proving, and viable argumentation (Stylianides, Stylianides, & Weber, 2017). The ECE framework offers a novel and promising way to do this. In fact, Hub and Dawkins (2018) called the ECE approach the only pedagogically useful approach to logical reasoning they could locate in the literature. In this article, we seek to contribute to the field by ascertaining whether a diverse group of U.S. Grade 8 mathematics students could learn to use the ECE framework in direct argument contexts, and by investigating the conceptions about mathematical claims and argumentation students developed in order to do so. We explore these students' understanding and usage of the ECE framework in what might be considered an ideal setting: the classroom teacher had emphasized viable argumentation regularly in her instruction. We examine the students' reasoning, in light of the ECE instruction the teacher conducted, to study what kinds of reasoning were revealed that supported and/or interfered with the students' use of the ECE framework. Through this exploration and analysis, we make suggestions for future research.

2. Background and framing

2.1. Classroom-based interventions in the area of proof in general

Stylianides et al. (2017) used the phrase *classroom-based interventions in the area of proof* to describe interventions designed to improve understanding or use of proof in formal mathematics learning settings at any grade level. Stylianides et al. (2017) wrote, "The number of such studies is small and acutely disproportionate to the number of studies that have documented problems of classroom practice [in the area of proof] for which solutions are sorely needed" (p. 253).

A handful of studies involving classroom-based intervention in the area of proof demonstrate that it is possible to shift students' perspectives on what constitutes proof in mathematics. Interventions have changed students' acceptances of empirical or intuitive justifications to desires for more secure justifications (Mariotti, 2013; Stylianides & Stylianides, 2009), improved students' understandings of norms for proof (e.g., Alibert & Thomas, 2002; Goos, 2014), and improved students' understandings that proofs depend on certain assumptions (Jahnke & Wambach, 2013). Creating environments where proving tasks are implemented can also improve students' understanding of mathematics concepts (e.g., Larsen & Zandieh, 2008; Weber, Maher, Powell, & Lee, 2008). Finally, explicit instruction about stating existence claims (including counterexample claims) and general claims can improve students' mathematical writing, to make this writing more akin to that of mathematicians (Yopp, 2015). Yopp demonstrated that explicit instruction on standards for generating and presenting arguments in response to a false general claim, and metamathematical practices like those in Lakatos (1976), improved the responses from elementary-school preservice teachers.

Despite the progress in the field, many interventions in the area of proof focused on the middle grades are either limited in their effect or lack details about the mechanisms used during the intervention, which are needed for researchers to build upon the interventions. For instance, Fan, Qi, Liu, Wang, and Lin (2017) reported on an intervention with Chinese Grade 8 students that involved instruction on drawing auxiliary lines to solve proving tasks. The researchers found no effect of the intervention for the treatment group over the control group. In another article in the same issue of *Educational Studies in Mathematics*, Fiallo and Gutierrez (2017) identified types of cognitive unity/rupture that induced students to produce proofs or produce empirical proofs in a case study of two students. Other than a general description of an environment where proof was emphasized and proof tasks were presented, Fiallo and Gutierrez did not present a particular intervention that would allow for the analysis of the relationship between the intervention and the resulting student reasoning.

In contrast, Stylianides and Stylianides (2009) provided an intervention with clear activities, instruction, intended outcomes, and mechanisms anticipated to cause the desired change in students' knowledge, thinking, or practices. Their tasks were designed to create cognitive conflict among preservice teachers (PSTs) about the role of empirical arguments in establishing a general claim. During intervention tasks, PSTs first accepted their empirical support for their general claims as "proofs," only to later find counterexamples among the examples not considered in their empirical data. This study was later replicated with high-achieving secondary students (Stylianides & Stylianides, 2014). Here, the intent was to create skepticism and an intellectual need for more secure arguments. The tasks were intended to cause mental conflict among students about the security of their empirical arguments. Student outcomes were measured against a standard for proof of a general claim that was explicit about what is not "proof"—that empirically testing a subset of cases in the domain of a claim is not secure/proof.

2.2. Background relevant to the eliminating counterexamples framework

The intervention on proof and proving that students in this study experienced, the Longitudinal Learning of Argumentation Methods for Adolescents [LLAMA] intervention, was grounded in prior research and perspectives on proving and viable argumentation roles and possibilities in mathematics learning environments (Yopp, 2011, 2014. 2015, 2017; Yopp & Ely, 2016). These perspectives are summarized here before the ECE framework itself is detailed.

2.2.1. The "rules of the game"

The LLAMA intervention reflects the idea expressed by Yopp (2015) that students need explicit instruction on the language of proof and the technical skills needed to construct viable argument proofs. For this reason, the intervention offered explicit instruction on the "rules of the game," meaning the canonical practices and beliefs of mathematicians. These rules of the game include using generic referents to support general claims and offering candidate mathematical objects with the desired properties to prove existence claims. Another rule of the game is about restricting claims to domains for which the argument actually provides viable support. Along these lines, Yopp (2015) found that students benefit from explicit instruction on the mathematical terminology register (e.g., *for all, if-then,* and *there exists*), about closed sentences (ones with no free variables, so that they are either always true or always false, e.g. 5 is a positive number) versus open sentences (ones for which truth depends on the values(s) of the variable(s), e.g. x is an integer), about counterexamples and their meanings, about the distinction between expressing one's ideas and thoughts and developing a viable argument or proof, and about the practice of exception-barring.

Studies have found that both students and teachers may believe that a general claim can be proven and yet also have counterexamples (Balacheff, 1991; Galbraith, 1981; Potari, Zachariades, & Zaslavsky, 2009; Stylianides & Al-Murani, 2010), although Stylianides and Al-Murani (2010) report finding no evidence of this misconception when they interviewed participants who at first appeared to express this view. Consequently, students may also need explicit instruction on the *law of the excluded middle* in propositional logic and its relationship to mathematical proof of a general claim. In mathematics, this law implies that a general claim is either true or its negation is true and is an existence claim. A proof either demonstrates that there are no counterexamples to a general claim or demonstrates that at least one counterexample exists.

2.2.2. Noncanonical and informal arguing

We recognize that students may express and acquire sophisticated ways of thinking about proof, proving, and viable argumentation that are not shared by mathematicians. These noncanonical ways of thinking may be leveraged to more sophisticated learning conceptions even if experts are unaware of them. To this end, we use Lobato, Hohensee, Rhodehamel, and Diamond's (2012) ideas about *pivotal intermediate conceptions* to support the development of the ECE framework and to ground it in pre-formal student reasoning. The idea is to leverage students' emerging ideas of proof, proving, and viable arguments when describing a framework that is accessible and useful to learners, with a vision of how these ideas could eventually lead naturally to formal canonical expert practice.

2.2.3. Reasoning schemes

The bedrock of the ECE framework to viable argument and proof as discussed in this article is that mathematical proof and proving can emerge among students differently from how expert mathematicians might envision them. We acknowledge that the mathematics education research community has little consensus on a definition of proof and proving (Weber, 2014). However, there are several characteristics of reasoning usually associated with proof, proving, and viable argumentation that the ECE framework emphasizes. One characteristic is that reasoning about a general claim involves a goal of establishing that a claim is true or false using some referent or model, whether mental or physical. This referent or model is used to represent all cases in the domain of the claim and expresses structural features (e.g., defining properties of the objects at hand) that can be manipulated to demonstrate the conclusion of the claim (or not the conditions when not the conclusion is addressed). Other characteristics include acknowledging and using prior results in inferences and being aware of meta-level justifications for the method or mode of argumentation employed (e.g., direct argument and proof by contradiction).

Spontaneous reasoning, perhaps without formal training, that can be considered viable has been the focus of numerous studies in the psychology literature. In particular, the mental models reasoning scheme (Johnson-Laird, 1983) asserts that reasoners engage in a sequence of stages involving representations analogous to the structure of the situation being reasoned about. In a *comprehension stage*, the reasoner constructs a mental model of the situation. Here, a *model* is a representation of the situation or its alternatives and can include examples, expressions, equations, diagrams, and descriptions. In a *description stage*, the reasoner develops a concise description of a model that asserts things not explicitly stated in the premises. In a *validation stage*, the reasoner searches for alternative models (e.g., counterexamples) that might refute assertions developed in a previous stage. If no alternative model is found, the assertions are taken as valid. Johnson-Laird (1983) summarizes this process as *eliminating the possibility of counterexamples*.

Mental models is a psychological scheme developed to understand people's reasoning about conditional claims, but we found the scheme useful as a theoretical basis as we developed the ECE framework. In many ways, the ECE framework is based on an analog of the mental models reasoning scheme. Reframed, mental models reasoning can be seen as a natural way of explaining more formal mathematical logic.

Suppose a reasoner is exploring a mathematical generalization in the form of a conditional claim *p(x) implies q(x) for all x*. From the perspective of mental models, that person concludes that claim is true if and only if no counterexamples exist. This biconditional inference is equivalent to any definition of proof of this general claim that attends to validity and soundness. A proof of *not q(x) implies not p(x) for all x* eliminates the possibility of the counterexamples *not q(x)* and *not (not p(x))*, which is *p(x)* and *not q(x)*. Therefore, a proof of *not q(x) implies not p(x)* (for all *x*) eliminates the possibility of counterexamples to the claim *p(x) implies q(x)* (for all *x*). A proof of *p(x) implies q(x) for all x* eliminates the possibility of the counterexamples *p(x)* and not *q(x)* directly. Either way, the reasoner has imagined counterexamples, which are models alternative to the generalized situation (claim). By arguing that these are impossible, reasoners are leveraging the validation act of mental models reasoning. In this way, the mental models research provides a natural way to ground students' reasoning about the validity of a general claim in their image of what the possible counterexamples to that claim would be and in their judgment about whether these counterexamples could or could not exist.

One issue with applying the mental models scheme to developing teaching and learning contexts for proof and viable argumentation is that we must ensure that students are viewing the claims we present them as quantified sentences that are either true or false. Durand-Guerrier (2003) pointed out that statements viewed as quantified (either true or false) by the teacher or researcher may be viewed as open sentences (true for some cases but not others) by students. Durand-Guerrier (2003) asserted that "natural" reasoning and conversational norms align more with open-sentence interpretations of conditional claims. Yopp (2015), on the other hand, demonstrated that students who initially responded "The conditional claim is sometimes true" in response to a false general claim could be taught to give responses that aligned with quantified interpretations of conditional claims and to develop two arguments, (1) a counterexample argument and (2) an argument for a general claim with a restricted domain.

2.2.4. Theoretical benefits of the eliminating counterexamples framework

The ECE framework offers several theoretical benefits. Yopp (2015) found that students can benefit from developing descriptions of all possible counterexamples to a general claim when participating in metamathematical practices similar to those described in

Lakatos (1976); i.e., when developing a mathematical theory about what is true, what is false, and what is uncertain). Furthermore, the mental models research shows that imagining counterexamples is a mode of thinking that people commonly engage in, and that people often conclude a general claim is true when they cannot imagine or find a counterexample. The ECE framework clearly builds on this reasoning by showing that a general claim is true when counterexamples have been entirely ruled out.

The ECE framework allows students to start investigating a claim without committing to a method of argumentation. Students eventually develop either a direct or an indirect argument without knowing ahead of time which one will be easier or best, since both types effectively can eliminate counterexamples.

Finally, the ECE framework allows students to understand how direct argument approaches prove a claim “P implies Q” true. Showing that all things with property P also have property Q shows that cases of P and not Q cannot occur, meaning counterexamples cannot exist.

3. Method

3.1. Context of the course

Instruction on the ECE framework occurred in the context of a larger LLAMA project/intervention. The LLAMA intervention was designed to make viable argumentation (e.g., proof) a practice through which students learned Grade 8 mathematics content. The broader LLAMA intervention was a significant part of the U.S. students’ regular Grade 8 mathematics classroom instruction through the entire school year, beginning in late August and ending in May. The ECE framework was a smaller part of the LLAMA intervention, although the bedrock of the intervention’s theory was about what makes an argument for a general claim viable or proof, and so it is impossible to separate the effects of the instruction about the ECE framework from the effects of LLAMA intervention as a whole. Consequently, this article is not directly a study of the effectiveness of the intervention or its parts. Rather, the focus is on the conceptions students develop about the ECE framework after receiving the intervention. In this section, we summarize the perspective, themes, and activities of the LLAMA intervention as presented to the classroom we examined. Below that, we describe in more detail the specific conceptual elements of the ECE framework the intervention served to develop for the students. Then in Section 3.3 we present the specific research questions for this study and in Sections 3.4-3.6 we describe the data we collected to study those research questions.

During LLAMA lessons, students were provided or were reminded of a formal mathematical definition or a quantified mathematical statement relating to the content to be learned and were asked to develop a variety of general or existence arguments in response to prompts. The students’ regular classroom teacher was trained in and agreed to adopt a practice of *teaching and learning with and through viable argumentation* in which mathematics concepts were taught and learned by participating in viable argument activities, including developing a viable argument for or against a mathematical claim and, at times, developing the claim. This teacher’s training included the ECE framework.

The LLAMA intervention was predicated on the notion that high-quality mathematics instruction supports viable argumentation as a daily feature of teaching and learning and a regular feature of assessment. This notion was aligned with the following summary of classroom practices.

To conform to the LLAMA notion of *teaching and learning with and through viable argumentation*, lessons about the content to be learned involved making or addressing general claims and existence claims about the content and supporting these claims. During these lessons, the teacher required students to be explicit about the domains to which their general claims applied. She also encouraged them to make these domains as general as they could, based on their data and their conceptual insights. She required students to use precise quantifying language when making existence claims (e.g., “there exists”) and general claims (e.g., “for all”) and to note when a general claim has a finite or infinite domain. Students and teachers were expected to be skeptical of empirical evidence (Brown, 2014; Stylianides & Stylianides, 2009) and to be conscious of whether they were generalizing based on patterns in their results or patterns in their processes (Harel, 2001). Students and teachers were expected to search for conceptual insights (Sandefur, Mason, Stylianides, & Watson, 2013; Yopp, 2015) that structurally link the conditions of the claim to the conclusion and explain why the claim holds for all cases. For example, when exploring general claims about lines cut by transversals and the angles created, students and teachers explored examples, equations, diagrams, and so forth, to find mathematical features that link properties of angle measures to features illustrated in a diagram that resulted from the properties of the lines. These properties invoked thinking about prior results.

For example, if two lines intersected, students might conjecture that a pair of interior angles created by the transversal on the same side of the transversal sum to less than 180 degrees. A diagram illustrating this conjecture might invoke thinking about a prior result about the sums of the measures of interior angles of a triangle, which provides a link between the properties of the lines and the angles created. This link provides a conceptual insight for proving the conjecture.

During the LLAMA intervention, students practiced argument approaches, including direct proving; showing every case of the conditions has the conclusion; creating logical chains, each implying logical necessities; contrapositive proving; contradiction proving; proving by exhausting all cases; constructive existence proving; and pragmatic (Cheng & Holyoak, 1985) and model-based (Johnson-Laird, 2005) reasoning that results in less formal arguments. The students experienced the notion of ECE as a way to access proof in these argument contexts, as detailed below in the ECE intervention.

These guiding perspectives and the perspectives expressed in the theoretical framing were organized into a collection of 12 conceptual pillars (CPs) for students to obtain. Table 1 summarizes these pillars. These pillars were organized into a learning progression by the researchers. The sequencing of the progression seemed reasonable to the researchers; however, we do not claim that

Table 1
Conceptual Pillars/Learning Progression of the LLAMA Intervention.

Conceptual pillar	Learning progression
1	Students conceive of viable argument as requiring explicitly stated features: a claim, a foundation, and a descriptive or explanatory link between the foundation and claim.
2	Students conceive of the mathematics register as communicating precise meanings. Students conceive of two types of claims in mathematics—general claims and existence claims—and they are acutely aware of the domain of the claims they present.
3	Students conceive of viable arguments for existence claims as providing an example in the domain of the claim and demonstrating that the example has the desired properties.
4	Students conceive of empirical arguments as insecure support for a general claim.
5	Students conceive of exhaustion as eliminating the possibility of counterexamples for general claims with finite domains.
6	Students conceive of proof as eliminating the possibility of counterexamples.
7	Students conceive of valid reasoning for general claims with infinite or large finite domains as applying viable logical reasoning schemas that eliminate the possibility of counterexamples.
8	Students conceive of referents as representative of all possible examples in the domain of a claim.
9	Students conceive of a viable argument for a general claim as requiring a conceptual insight that applies to all possible examples in the domain of a claim.
10	Students conceive of a viable argument for a general claim as appealing to and using prior results.
11	Students conceive of an indirect argument for a general claim as viable because it eliminates the possibility of counterexamples.
12	Students conceive of viable argumentation activities as requiring a decision about what mode of argument to use (e.g., exhaustion argument, existence argument, etc.).

the particular sequence is essential, and perhaps students could acquire all 12 conceptions in another ordering.

3.2. Context of student participants

3.2.1. School-wide practices that supported success in LLAMA framework

While the students in this study represent a typical range of Grade 8 mathematics learners, their school culture was not typical of American classrooms. This small K – 8 school emphasizes depth over breadth at all levels, which was embodied in a culture of "grappling" and thinking deeply about every subject. Across subjects, students were encouraged to use evidence from text in support of informal arguments. Students also received explicit instruction on the eight Common Core State Standards mathematical practices (NGACBP & CCSSO, 2010). Included in the overarching vision of the school was the notion that learning is challenging. Students at all levels were pushed and supported to do more than they thought they could. There was a school-wide emphasis on justification of thinking, and students were often asked, "Why do you think that?" "Why does that work?" "How do you know?" and "Will that work or be the case every time?" In mathematics, students at all levels were expected to make arguments for their solutions; to justify their reasoning using numbers, words, graphs, and diagrams; and to question and critique others' reasoning. However, outside of the LLAMA lessons, these justifications were not necessarily based in canonical mathematical methods and explicit standards for proof, proving, and viable argumentation.

3.2.2. Classroom practices that supported success in LLAMA framework

These students worked with the same mathematics teacher for Grades 7 and 8. In Grade 7, the teacher formally introduced them to the practices of generalization, justification, and employing mathematical properties and definitions to justify their thinking. This work continued through Grade 8. Early in the year of this study, students were taught that to prove a general claim with a large domain, no amount of empirical evidence is sufficient (i.e., LLAMA, CP 4). They were taught that every case in the domain would need to be verified, and that is not possible for a sizable domain. Instead, logic was required for such proof/viable argumentation. Later in the year, students were taught that mathematical definitions, properties, and prior results must be used logically to guarantee that a given claim is true (i.e., LLAMA CPs 5–10). Students were taught that because a single counterexample proves a general claim false, to prove a claim true the argument must eliminate the possibility of counterexamples (CPs 6 and 7).

3.2.3. Instruction pertaining to the ECE framework

Instruction directly pertaining to developing the ideas in the ECE framework comprised roughly three hours of class time focused on developing ideas of CP 6 and CP 7 and occurred prior to the student interviews. Earlier in the year, students participated in two skepticism lessons in which they checked numerous confirming cases to a general claim and became convinced the claim was true, only to discover or be shown a somewhat obscure and non-confirming case that proved the claim false. These experiences led to a healthy skepticism of empirical evidence as the sole support of a general claim with an inexhaustible domain. We mention this activity not because it was part of the ECE intervention but because, based on students' responses during the interview, these activities may have influenced students' reasoning.

We hypothesized students who received instruction on the ECE framework for general claims would display understanding of the following intermediate conceptions (ICs), modified from those in Yopp (2017):

IC1: [Definition of counterexample] A mathematical object is a counterexample to a conditional claim if and only if the object meets the conditions of the claim and does not meet the conclusion of the claim. Consequently, the collection of all possible

	THEN part (conclusion)		
IF part (condition)		Conclusion met	Conclusion not met
	Condition met		
	Condition not met		

Fig. 1. Table for constructing descriptions of conforming, nonconforming, and irrelevant cases.

counterexamples to a conditional claim is characterized by the properties *meets the conditions of the claim* and *does not meet the conclusion of the claim*.
IC2: [Truth criteria for general claims] Conditional claims are true if and only if counterexamples to the claim do not exist (are impossible). Consequently, general claims are false if and only if there exists a counterexample.
IC3: [Describability of counterexamples] The collections of all possible counterexamples to a conditional claim can be described by their properties, even when counterexamples are impossible.
IC4:[Direct argument viability condition] Descriptions of all possible counterexamples to a conditional claim can be useful in evaluating arguments. Arguments for general claims are viable (or proof) if and only if counterexamples are eliminated by showing every case of the conditions also has the properties of the conclusion.

We anticipate that these ICs may often, although not necessarily, develop in this order. To develop these conceptions, instruction on the ECE framework included two key elements: (1) developing descriptions of all possible counterexamples to a general claim and (2) proving a general claim true by eliminating counterexamples.

3.2.4. Instruction on conditional claims, examples, nonexamples, and counterexamples

The teacher used the Wason card selection task (Johnson-Laird & Wason, 1970) to introduce a table structure for conditions and conclusion of a general conditional claim (see Fig. 1). Students practiced completing the table using nonmathematical claims, such as "If I have a beagle, then it is a dog" and "If I have a dog, then it is a beagle," and then extended this work to mathematical claims such as "If a number is a multiple of 12, it is also a multiple of 4," its converse "If a number is a multiple of 4, it is also a multiple of 12," and "Any pair of lines with different slopes will intersect." After discussing all four of the types of cases in the boxes, the students discussed which cases would show that the claim is false. They discerned that cases in the bottom row would not serve this purpose, because the claim does not apply to these cases, since the claim's conditions are not met. Students learned that it was only the cases in the upper right cell, which meet the claim's conditions but *not* the conclusion, that show a claim is false. These cases were then called counterexamples. The teacher explained that since a general claim must be true for all items in the domain of the claim (that meet the conditions of the claim), if even one counterexample to the claim exists, the claim is considered to be false. The teacher also confirmed that items not meeting the claim's conditions (nonexamples) are irrelevant to the claim and do not bear upon whether the claim is true or false. Fig. 2 illustrates some of our sample students' responses.

3.2.5. Proving a general claim true by eliminating counterexamples

Instruction on the ECE framework began with claims from everyday life, such as "All mammals live on land." Students identified whales and dolphins as mammals that do not live on land and recognized that these animals served as counterexamples, proving the claim false. The teacher asked students to consider the role of counterexamples in relation to proving a claim true. She said,

> This claim is false because we can find a counterexample. What would we have to do with this section [points to upper right cell in the conforming examples, nonexamples, and counterexamples table]—this conditions met, conclusion not met—to prove that a claim is true? What do we have to do in order to prove that a claim is true, a general claim is true, if what we are saying is if I can show a counterexample, then I am proving that the claim is false?

After some conversation, multiple students came to the conclusion that to prove that every single unique mammal lives on land, you have to prove that there is no possibility of a counterexample. The teacher reiterated this point, saying,

> Actually, we have to prove that it is *impossible* for something to be in that box. It is not like we were just not creative enough or we forgot about something. . . just *nothing could exist* in that box. If we can find a counterexample, we can prove that a general claim is false. If we can eliminate the possibility of a counterexample, then we have a really strong argument to show that our claim is true.

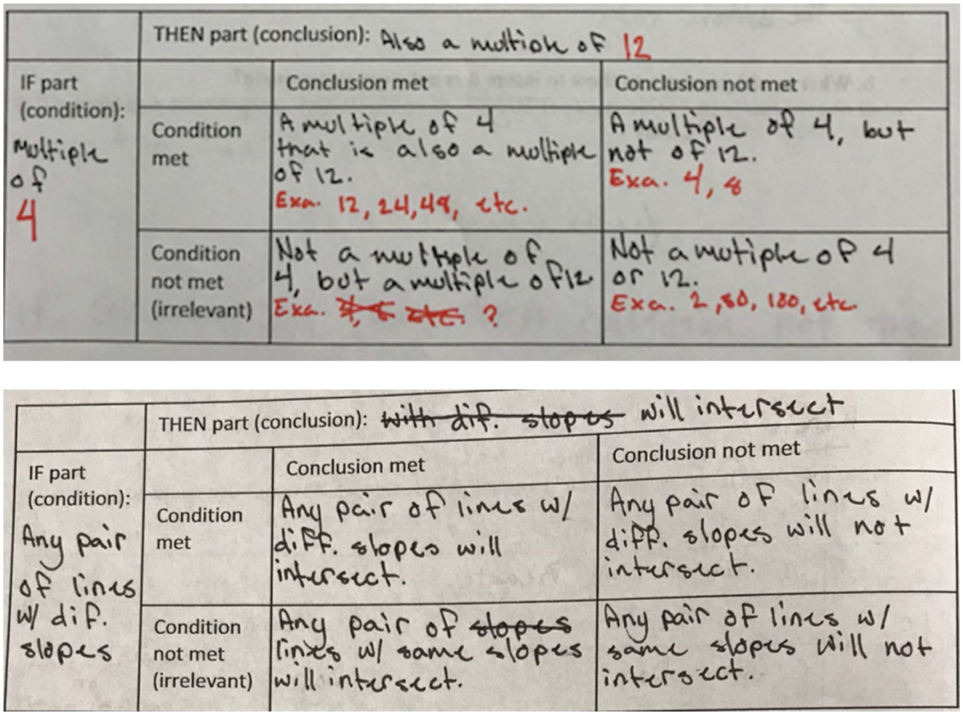

Fig. 2. Two students' work on completing tables with descriptions of conforming examples (upper left), nonexamples (bottom row), and counterexamples (upper right).

Students next completed examples, nonexamples, and counterexamples tables to create viable arguments for or against the following two mathematics claims, identifying a counterexample for a false claim and creating direct arguments for a true claim:

1 If a number is a multiple of 12, then that number is a multiple of 4.
2 If a, b, and c are natural numbers and $b > c$, then $a^b > a^c$.

For Claim 1, students filled in entries in the table for examples, nonexamples, and counterexamples tables; they found that they could describe the class of counterexamples (multiples of 12 that are not multiples of 4), even though they could not find any specific counterexamples. As a class, they established that the claim is true because 4 is a factor of 12. The students noted, in their words, that "anything that 12 goes into, 4 goes into also." Hence no counterexamples to the claim could possibly exist.

For Claim 2, students described examples, nonexamples, and counterexamples and identified the case $a = 1$. They explained that the case $a = 1$ provides a counterexample. The teacher then asked the students to modify the claim to make it true. Most students changed the claim's domain, adding a new requirement that $a > 1$. Students then argued for the revised claim. One student presented the example $a = 2$, explaining that 2·2·2 will be greater than 2·2 because multiplying more copies of a natural number (i.e., more multiples of the base) creates a greater number. This student attempted to generalize this example, asserting that any natural number base and any pair of unequal natural number exponents will behave similarly—the number with the lesser exponent will be less than that with the same base to a greater power.

The teachers leveraged these students' informal arguments to lead the class in a discussion of how the mathematical definitions of exponents, multiples, and multiplication, and the properties of exponents, which the students' arguments implicitly referenced, can serve to eliminate the possibility of counterexamples to the claims. She referred to the examples, nonexamples, and counterexamples tables and described how the general reasoning the students provided *guarantees* that every case in the domain (top row) cannot fail the conclusion and so cannot be in the right column. In order to make this guarantee clear, she explained, it was important that the properties and definitions used in their arguments were clear.

3.3. Research questions

We developed the following research questions regarding the students' argumentation after students received the instruction described in the previous sections:

1 For students instructed in the ECE framework, what viable argumentation conceptions and subconceptions can be identified in

Table 2
Past Mathematics Achievement Levels of the Sample Students.

Name (pseudonym)	2015/2016 achievement level	2016/2017 achievement level
Elly	Advanced	Advanced
Olive	Advanced	Advanced
Chad	Advanced	Advanced
Jill	Advanced	Advanced
Ken	Advanced	Advanced
Mia	Proficient	Proficient
Bo	Proficient	Proficient
Tengu	Proficient	Advanced
Zach	Basic	Proficient
Anton	Basic	Proficient

their reasoning in response to a direct argument task?

2 Which of these conceptions and subconceptions support LLAMA's vision for viable argumentation and the ECE framework, and which serve as obstacles?

3.4. Sample

The sample included 10 students enrolled in Grade 8 mathematics who volunteered to participate in periodic interviews throughout the school year. They represented a diverse level of past academic achievement in mathematics, although students scoring at the advanced level were overrepresented. These achievement levels were measured by annual mathematics achievement assessments, using the Smarter Balance Assessment Consortium grade-level assessments (http://www.smarterbalanced.org/). Table 2 illustrates the achievement levels of sample students.

The study took place at a charter school in the northwestern United States. In the United States, charter schools are publicly funded but are managed by independent not-for-profit organizations and may have limited resources. They follow the same rules and regulations as public schools and participate in the same state-mandated assessments.

3.5. Data

The instruction on the ECE framework occurred in early February, and data were collected in mid-February using a two-part semi-structured task-based interview. In Prompt 1, participants were asked to consider the claim *If you multiply two square numbers, you will get another square number,* to determine what properties a counterexample to the claim would have, and to provide a written response with their ideas. They were provided with the table in Fig. 1, which they had used previously for the purpose of identifying properties of counterexamples.

Once students completed this part of the task, they were asked probing questions, such as "Do you think the claim is true or false?" "Can you think of any counterexamples to the claim?" and "Can you make an argument for the claim?"

In Prompt 2 of the interview, students were presented with an argument for this same claim and were asked to determine if the argument eliminated the possibility of there being any counterexamples and to explain why or why not. The argument is given below.

Claim: If you multiply two square numbers, you will get another square number.
Argument: Say you have two square numbers. That means you can write them as a^2 and b^2.
Their product is a^2b^2
$= (a \cdot a) \cdot (b \cdot b)$
$= a \cdot b \cdot a \cdot b$
$= (a \bullet b)^2$.
Since the product can be written as something squared, that makes it a square number too.

After writing their responses, students were asked probing questions such as "Do you understand the argument?" "Do you think the argument is viable? Why or why not?" and "Do you think the argument eliminates the possibility of their being a counterexample to the claim?" They were allowed to modify their written response at any point in the interview. Interviews were video recorded and later transcribed, and student written work was collected.

3.6. Data analysis

Data were first analyzed by the researcher who conducted the interview. The first four authors each interviewed two or more students. Each researcher then drafted a summary of the students' responses relative to the ECE framework, focusing on student comments and writings that conformed with or diverged from anticipated responses. These summaries drew on and integrated student responses to both Prompts 1 and 2 and included example student comments from the transcriptions and pictures of student work to illustrate and support the researcher's analysis.

The summaries were presented to the first author as the research lead. The first author then read the summaries and reviewed the transcripts and copies of student work to validate the analysis. The summaries were organized into groups of similar responses. The data were then reanalyzed using the ICs mentioned above as a framework. Each student's data were analyzed for evidence of whether or not each IC was expressed, using a constant comparison approach in which the emerging descriptions of the students' data were compared to the ECE framework and the ICs until descriptions of students' thinking and activities emerged and stabilized. Ultimately, descriptions of students' thinking and activity were organized into those that conformed with our intended instructional goals (using the ECE framework as theorized), as expressed through the ICs, and those that diverged from our intended instructional goals. Students whose thinking diverged from our instructional goals were then organized into the groups in which the divergent thinking appeared to support an understanding of the ECE framework, perhaps in ways we had not envisioned, and groups in which thinking diverged from the ECE framework and was perhaps a barrier to acquiring conceptions consistent with our instructional goals. Through this process, we were able to elaborate on the ICs and identify conceptions within the ICs that we had not anticipated, which we call "sub-ICs," or subconceptions. For coding purposes these subconceptions we identified are each necessary, and together they are sufficient, for the student to be coded as displaying the broader IC. In other words, we coded a student with IC3 if and only if they displayed evidence of IC3a-c.

4. Findings

All students are described using pseudonyms. Nine of the 10 students leveraged the ECE framework in some manner and usefully brought the framework to bear when assessing the viability of the argument provided. Four of the students—Jill, Elly, Chad, and Olive—demonstrated correct and robust understandings of the ECE framework and used the framework explicitly to evaluate the viability of the argument. Five of the students—Ken, Tengu, Zach, Mia, and Bo—showed significant understanding of the ECE framework but displayed some obstacles to using it effectively. One student, Anton, showed little understanding. In this section we detail these students' reasoning in order to highlight particular conceptions of the ECE framework that emerged through our analysis.

4.1. Intermediate conceptions 1–3

Most of the students in our sample explicitly articulated an understanding of IC1 and IC3 in response to Prompt 1. Most showed understanding of IC2 as well, although not always as explicitly. We detail these understandings in this subsection.

Eight of the 10 students—all except Zach and Anton—clearly articulated an understanding of IC1. Here are a few representative examples of comments they made indicating this understanding:

- Elly: "[A counterexample is] when the condition's met and the conclusion is not met."
- Ken: "Basically a counterexample is when the 'if' part, the condition is present, is met, but then the conclusion isn't. So the counterexample would be when you multiply two square numbers, so the condition is met, but then you don't get another square number."
- Tengu: "A counterexample would meet the conditions but not the conclusion of a claim." Tengu added "that the conditions of this claim were that two square numbers multiplied together, and that the conclusion of the claim is that you would get another perfect square," and wrote, "a counterexample would be a 'not perfect square' that you could get by multiplying two squares together."

Eight of the 10 students—all except Bo and Anton—articulated a clear understanding of IC3. Bo showed a nearly complete understanding, but we later discovered he lacked a subconception, which we had not anticipated. This will be described below. All eight accurately described the properties that a counterexample to the given claim would have, as Ken and Tengu's quotes show in the previous paragraph.

Most students used the examples, nonexamples, and counterexamples table to reason about what counterexamples to the claim would look like. For example, Chad wrote that "a counterexample would be if you multiplied two squares and didn't get another square" and then filled in the table (see Fig. 3), offering examples in three of the bins. He described how each of the cases in the four boxes in the table bore upon the claim: "Everything else wouldn't apply," pointing to the bins in the bottom row, "or [would] be in this box," pointing to the box for conditions and conclusions met. "Neither of these boxes [bins in the last row] apply because if it doesn't have the if part, the conditions, then it doesn't matter if you get the conclusion. . . but if it gets the *if* but not the *then*, that's when it's a counterexample [italics added]." Chad then stated that he thinks there is no counterexample to the claim but also acknowledged that his empirical work did not prove this assertion.

Based on the students' responses, we found that IC3 actually contains several distinct elements. These elements we discerned and labeled IC3a, IC3b, and IC3c.

IC3a:Counterexamples to a claim can be described in terms of their properties (meets the condition but not the conclusion).

Nine of the students articulated this particular combination of properties for this claim, as Chad did above, "A counterexample would be if you multiplied two squares and didn't get another square". The 10th student, Anton, instead did not consistently treat counterexamples (or other kinds of examples) as objects simultaneously displaying a pair of properties. He did not use words like "and" or "but" to connect the properties of "meets the conditions" and "meets (or does not meet) the conclusion." Rather he used

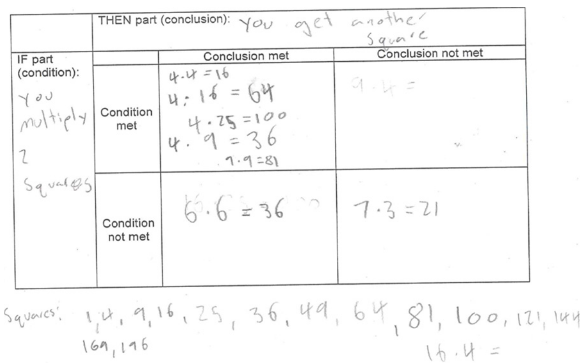

Fig. 3. Chad's written work in the examples, nonexamples, and counterexamples table.

conditional language like "if," expressing directional thinking, as if the conditions or conclusions must be met first before the other part can be discussed or negated. Anton's case underscores the importance of IC3 in our situation. As will be demonstrated shortly, Anton's inconsistent description of potential counterexamples interfered with his ability to apply the ECE framework to assess the argument in Prompt 2.

IC3b: The cases with the properties in IC3a comprise all of the counterexamples to the claim.

Eight of these students explicitly said that the cases satisfying the conditions and not the conclusions represented *all* the counterexamples to the claim. For instance:

- Ken: "In my saying that I have two square numbers but when multiplied they don't equal another square number, yeah, I think that that encompasses all counterexamples."
- Elly:

Interviewer: How do you know this [pointed to her description] describes counterexamples?
Elly: Because the condition's met and the conclusion is not met.
Interviewer: Does your description describe all counterexamples or just some of them?
Elly: The description is all counterexamples, I think.
Interviewer: Why do you think it is all counterexamples?
Elly: Because this is opposite of the conclusion and this is the conditions [pointed to the respective parts of her description], so the condition's met and the conclusion is not met.

Yet this idea of *all* counterexamples is distinct from IC3a, because one student, Bo, did not display this understanding even though he understood IC3a. He wrote, "A counterexample would need to show that you could multiply two square numbers and get a number that is not a square number, showing the claim doesn't work." But when asked if this represented all possible counterexamples, Bo said there might be some others. He was uncertain how these "others" might be different from those he described, but he was unwilling to say they didn't exist. This response motivated us to construct another subconception that might be important to a complete understanding of IC3:

IC3c: Potential counterexamples may be hypothetical entities, especially if the claim is true.

Bo had a speculative notion about unrealized or abstractly conceived counterexamples, but Bo lacked the notion of counterexamples as hypothetical entities as we envisioned them. For Bo, counterexamples are speculative entities in that, for the sake of argument, these unidentified objects might be out there, somewhere. Yet, to use the ECE framework to validate a viable argument for a general claim, the idea of potential counterexamples is entertained as a way of assessing whether or not these hypothetical objects have been eliminated. For the ECE framework to be effective, these hypothetical objects must be characterized by their properties. As stated in IC3a, the student pins down the exact properties of these hypothetical entities to determine whether or not their existence has been eliminated. Consequently, IC3c can potentially serve as either an affordance or a barrier to using the ECE framework when validating a viable argument for a true mathematical general claim depending on whether IC3a and IC3b are in place.

In fact, most of the students in our study believed the general claim we gave them to be true and stated that there were no counterexamples to it (IC2). Yet these students could articulate what the properties of counterexamples *would* be. Many students used the subjunctive mood to describe this, such as Jill's use of "would have" when she said, "A counterexample would have two perfect squares multiplied together and have a product of a non-perfect square." It may also be that students were mirroring the interviewers, who also tended to use this terminology, but we assert that the subjunctive mood expresses students' awareness that eliminating counterexamples involves entertaining the notion of possible counterexamples, even when they believe counterexamples do not exist.

Focus on whether or not an argument eliminated counterexamples also promoted students to focus on the domain of the claim. This is because they were imagining examples that satisfied the claim's conditions (i.e., were in the domain) and not the conclusions. For instance, Mia and Jill critiqued the argument in Prompt 2 for not explicitly defining the domain of the variables a and b. Mia asked the interviewer to clarify whether or not a and b could take on values other than natural numbers. If so, Mia noted, the argument would not apply to the claim/situation. Jill wondered if the argument was careful enough in excluding decimals as substitutions for a and b. We attribute this focus for Jill and Mia to their explicit attention to whether counterexamples were eliminated in the argument and to their recognition that counterexamples must be in the claim's domain.

Most of the students demonstrated an understanding of IC2, although students' understanding of IC2 was not always as explicit as it was for IC1 and IC3. For instance, even though Bo was not certain that the only counterexamples were the cases that satisfied the conditions and not the conclusion, he *was* certain that the claim would be true only if no counterexamples existed. He also said the claim would be false if a counterexample did exist. Elly found a (false) counterexample based on a miscalculation and explicitly concluded that the claim was false. Later, she corrected herself, and those details are below.

4.2. IC4 and related subconceptions

Some of these students' comments revealed subconceptions associated with IC4 that we had not previously recognized. To analyze these conceptions, we found it useful to distinguish between the two parts of IC4:

IC4a: Descriptions of all possible counterexamples to a conditional claim can be useful in evaluating arguments.
IC4b: [Direct] Arguments for general claims are viable (or proof) if and only if counterexamples are eliminated by showing every case of the conditions also has the properties of the conclusion.

We have added [Direct] to IC4b to emphasize that this IC is specific to direct argument contexts while the other ICs can be applied to any argument for a general claim.

In response to Prompt 2, four of the students—Olive, Chad, Jill, and Elly—demonstrated a very robust understanding of IC4, although IC4a was sometimes tacit. Three more students—Mia, Ken, and Bo—showed strong but not entirely complete understanding of IC4. Tengu and Zach used elements of the ECE framework but had more difficulty with the argument in Prompt 2, and Anton did not consistently use the ECE framework.

Chad's responses to Prompt 2 provides a typical example of the thinking of students who demonstrated an understanding of IC4b. Chad wrote, "This argument is viable because it gave a proof of why any two squares multiplied will be a square. In this argument, they used the rule that $a^2 \cdot b^2 = (a \cdot b)^2$ to prove their claim." The interviewer asked him, "Do you think the argument eliminates the possibility of there being a counterexample to the claim? If so, explain." Chad replied:

> Yes, because it shows how if you'd like. . . when it says *a* squared times *b* squared equals *a* times *b* squared then it says that there won't be. . . there will never be a time when you can multiply two squares and not get another square.

This comment and his other responses reflected Chad's understanding of IC4b; yet Chad's comments also illustrated IC4a because Chad refers to a description of counterexamples in explaining why such objects cannot exist. He explained why it is the same thing to show that the general claim is true via direct algebraic transformations as it is to show that there will never be any counterexamples (that is, cases where the conditions are met but the conclusion is not).

Student understanding of IC4a was more difficult to directly assess because students may not be explicit about what was useful to them when addressing mathematics tasks. For IC4a to be revealed, students needed to explicitly refer to a potential counterexample or a general description of counterexamples when discussing the viability of an argument. One student, Elly, actually referred directly to a (false) counterexample that she developed in response to Prompt 1 when evaluating the argument presented in Prompt 2. Her responses were particularly interesting in illustrating how IC4a can be intertwined with activities associated with IC4b.

In response to Prompt 1, Elly wrote the response presented in Fig. 4. Based on her incorrect calculations, Elly treated her work as a counterexample argument to the claim presented in Prompt 1. She said that the claim was false.

For this reason, when Elly read the argument presented in Prompt 2, she initially wrote that the argument was not viable (see Fig. 5), but then, as she continued to examine it, she paused and then crossed out her writing. She said, "I messed up my multiplication. . . 4 times 9 is not equal to 45." She then worked through another example, shown in Fig. 5, which she described as "putting numbers in for the variables" in the argument provided in Prompt 2. She then reconsidered the argument and said that it was actually viable. The interview continued with the following exchange:

Interviewer: Why do you believe this argument is viable?

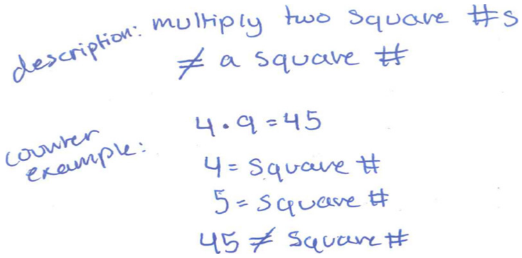

Fig. 4. Elly's incorrect counterexample to the claim in Prompt 1.

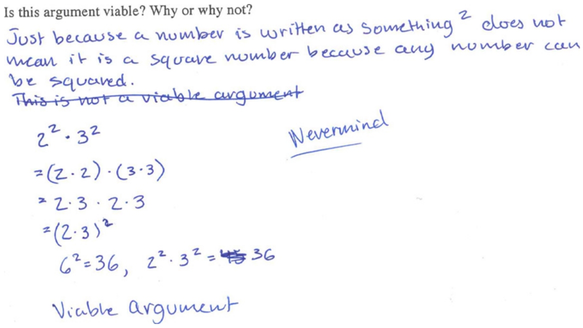

Fig. 5. Elly's corrections to the incorrect counterexample Elly constructed.

Elly: Because since it's variables so it's applying to all square number so there is not any room for non-examples, I think.
Interviewer: Do you think the argument eliminates the possibility of counterexamples?
Elly: Yes? I think so. . . Because it covers everything.
Interviewer: What did it do for your example?
Elly: It proved it wrong.

Elly's reasoning reflects how the ECE framework was accessible and useful for her when working with a direct argument for a general claim. As she examined the direct argument, she realized that it eliminated her own counterexample from earlier. This caused her to check her calculations and discover her error. She explicitly said that this argument eliminated the possibility of all counterexamples, including the one she had mistakenly come up with earlier. For Elly, viable arguments for general claims and counterexamples to general claims cannot coexist.

4.3. Algebraic representation and manipulation

In contrast to these students, Tengu seemed unwilling to recognize the argument in Prompt 2 as being entirely viable. He wrote:

> I think that this argument could be part of a viable argument but by itself doesn't prove that two square numbers multiplied will get another square. It proves that you can take two square numbers and multiply them to get another square number, but it does not prove that you cannot have a counterexample.

Tengu explained further that he thought it was *likely* that the claim was true and that there were no counterexamples, but that he did not trust the algebraic argument as showing that a square number resulted for every case of the condition. Tengu said vaguely, "They could prove it a bit better."

The main reason for Tengu's distrust for the argument appeared to be because he did not see how the algebraic symbolism effectively accounted for all possible cases. Indeed, there is strong evidence that he had a solid theoretical understanding of IC4. For instance, when pressed by the interviewer about what it would take to prove that you can't have a counterexample to conditional claim, Tengu said "it would have to prove not that it *can* be a square number, but that it *has* to be and that there is no other way to get a non-square number from two square numbers." He also displayed understanding of IC1–IC3 and could describe the ECE framework generally, including describing what it would take to prove that counterexamples to a general claim cannot exist. Yet algebraic representations and transformations here appeared to serve as an obstacle to his full application of the ECE framework.

Zach and Anton also were uncertain about the algebraic representation and manipulation in Prompt 2 and were not sure that the variables represented all cases in the claim's domain. This uncertainty about variables representing generality interfered with their application of the ECE framework. These students' responses indicate that a prerequisite to using the ECE framework for direct variable arguments is that students must understand the role of variables as placeholders for all cases in the domain of the claim and know that manipulations of these variables represent manipulations for all cases in the domain of the claim. We note that this knowledge is likely a prerequisite for anyone who would assess general direct arguments with variables, not just for someone using the ECE framework.

4.4. Overgeneralizing skepticism

Mia was inconsistent in her account of whether the argument in Prompt 2 eliminated counterexamples. At first she said that it did, but then later she hedged her bets and said that there might be other "unimportant" counterexamples out there somewhere. Tengu, Anton, and Zach also expressed skepticism about whether all counterexamples can indeed be eliminated for a general claim with an infinite domain (Zach's case is examined more closely below). These students' skepticism may be an unintended consequence of two skepticism lessons that were part of our broader intervention. In these lessons, the teacher presented a general claim that appeared to be true after empirical investigations but was later proven false with a counterexample not found in students' empirical work. These activities may have induced for some students a deeper skepticism of all general claims with infinite domains regardless of the argument. This hypothesis is supported by the fact that Anton explicitly critiqued the general claim in Prompt 2 as being "too broad"

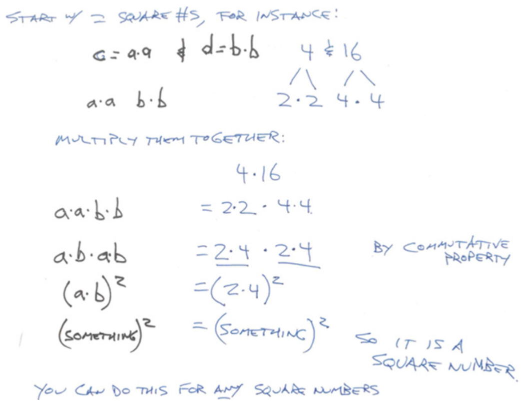

Fig. 6. The interviewer's instantiations of the argument presented in Prompt 2, with the interviewer's explanations for various steps in the argument, as presented to Zach.

and suggested that the claim's domain be restricted.

4.5. Falsifiability

Like Mia and Tengu, Zach also described skepticism about the argument eliminating all counterexamples. But Zach articulated an interesting reason for this that warrants its own category of conception. It took a while to reveal it, because Zach changed his mind several times about the argument's viability. At first he did not understand key features of the argument. For instance, he thought the argument began with a and b, not a^2 and b^2. He also initially did not acknowledge the relational structure in the argument that transformed the algebraic expressions across the equal signs. Later, he said that the argument is not viable "unless you're going to use the commutative property." He acknowledged that the algebraic transformations "might work" but that it "depends on your perspective."

To explore what Zach meant by all of this, the interviewer presented an instantiation of the argument in a generic example argument, in an attempt to clarify and focus the student on processes in the argument and structural relations the processes present (see Fig. 6). After this, Zach still was unsure whether counterexamples were ruled out, so the interviewer rewrote the general variable argument alongside the generic example argument, line by line, so Zach could see how the two were in parallel. In Fig. 6, the generic example argument is on the right and the algebraic argument is on the left.

At this point Zach had seen the argument in many forms, and it appeared that his particular questions and objections had all been addressed. Then the following exchange occurred:

> Interviewer: Now, do you think that if we did it this way and were really clear, would *that* eliminate the possibility of counterexamples or not?
> Zach: As long as a and a [are] equal, when they're times one another, they equal a, um, square number, then, and *b*and *b* is the same thing, then it should potentially work.
> Interviewer:. . . What you were saying before, with the original claim, was that a counterexample was something where you started with two square numbers, and multiplied them, and got something that's *not* square. And this, you're saying, would rule that out?
> Zach: Yeah, because if you have two. . . well, it wouldn't rule it out. It would just. . . it would give the generalization for a rule that, uh, until disproved, then, uh, it's right.
> Interviewer: Tell me more what you mean.
> Zach: Uh, so because see, if. . . *a* times *a* does equal a square number, but I don't necessarily know, um, with limited knowledge I have, um, with the fact, um, if every single square number that exists times another square number that exists, would equal another square number that exists. I have a pretty good idea of how that would work, which, I'm pretty sure it would all work. But I don't know how, if it would all work. I'm just making a guess. So until someone came up with a—something to disprove this, like a square number times another square number that doesn't equal a square number, then this claim is correct, and in order to find that, I don't have the knowledge for it.

Zach is doing more than just hedging his bets here. He seems to be applying to this claim a Popper-like falsifiability criterion. Popper's (1963) view was that a scientific hypothesis must be *falsifiable*—it can never be proven true, but it must always be possible to conceive of an experiment that could prove it false. Thus, we can never be sure a scientific hypothesis is true; we can at best say it has not yet been falsified. Zach is saying that he can never be certain that this claim is true because counterexamples that he has *yet* to encounter may be out there waiting to be discovered. This might be due to his limited knowledge of the role of variables and algebraic manipulations, but for Zach this argument, and perhaps every argument, has no power to establish the claim as perpetually true.

This interpretation accords with comments Zach made earlier in the interview, when he expressed that he liked counterexamples more than situations where he could not find them: “When something actually doesn’t have a counterexample it’s just, it’s very irritating.” When the interviewer asked him why, Zach replied:

> Well, I mean, you’d want to be right, but I feel like it’s a lot more fun trying to find a counterexample. And when you’ve found it, or when you find it, it’s just like, all right, this is good. Good job. But when you’re trying to find it and it’s just not there, it’s just like. . . [in an exasperated tone] “eh, why!” [the student made a scribbling motion].

It may be that existence arguments are more transparent to Zach than general arguments, because in the latter the representations must stand for cases not necessarily envisioned (see Zazkis & Liljedahl, 2004, for a discussion of transparent and opaque representations).

Zach’s idea of falsifiability is not the same as the skepticism discussed in the previous section. We used the term “skepticism” for distrust of a particular argument for a general claim when the argument was not seen as accounting for all possible counterexamples. Zach appears to doubt whether *any* argument for a general claim could be viewed as unassailable. This distrust may relate to a preference for demonstration proofs for existence claims, which are arguably more tangible than general arguments.

In fact, Zach’s falsifiability idea conflicts with the ECE framework and appears to serve as an obstacle to understanding the framework. Skepticism alone may not present the same obstacle. Some students identified as overly skeptical were worried about the particular argument’s ability to address the claim’s infinite domain. Zach’s concerns, and perhaps Mia’s, involved unforeseen counterexamples not accounted for in the general descriptions provided in response to Prompt 1. Together the two constructs, skepticism and falsifiability, highlight the importance of another meta-understanding that is crucial to the ECE framework, and without which the framework cannot be coherently used to interpret and assess direct arguments. This meta-understanding is that *it is possible to actually eliminate all counterexamples with a direct general argument*. Put another way, for a claim “If P, then Q,” showing that every object satisfying the condition P must also satisfy the conclusion Q actually serves to eliminate the possibility of counterexamples. Yet it is plausible that Zach’s falsifiability idea serves as an obstacle to his acceptance of direct general arguments no matter what intervention approach is taken, ECE or otherwise. For Zach, a viable argument for a general claim can coexist with counterexamples to that claim. It is also noteworthy that Zach is the only student in the group who clearly lacked this crucial meta-understanding.

4.6. Summary of findings

Here, we summarize our findings with respect to the research questions. Nine of the 10 students readily accessed the ECE framework and expressed a general understanding of the framework. These nine students also used the ECE framework successfully when discussing and assessing the viability of the direct argument for a general claim presented to them. Four of them demonstrated robust understandings and used the framework when assessing the argument. The other five showed significant understandings of the ECE framework but displayed some conceptions that interacted with or interfered with its use. Table 3 summarizes these findings.

By analyzing these students’ comments and activities, we discerned or affirmed a number of conceptions and subconceptions that supported, and some that served as obstacles to, the ECE framework and how these affected the students’ ability to analyze the direct argument. Of these, we labeled and tracked the supporting conceptions, because these contribute to our broader goal of developing a thorough account of how students gain an understanding of viable argumentation as described in our intervention and the ECE framework:

> IC1: A mathematical object is a counterexample to a conditional claim if and only if the object meets the conditions of the claim and does not meet the conclusion of the claim. Consequently, the collection of all possible counterexamples to a conditional claim is characterized by the properties *meets the conditions of the claim* and *does not meet the conclusion of the claim*.
> IC2: Conditional claims are true if and only if counterexamples to the claim do not exist (are impossible). Consequently, general claims are false if and only if there exists a counterexample.

Table 3
Intermediate Conceptions Presented by the Students.

	IC1	IC2	IC3a	IC3b	IC3c	IC4a	IC4b
Anton		T	N	N	Y	T	T
Bo	N	T	Y	N	Y		
Chad	Y	Y	Y	Y	Y		Y
Elly	Y	Y	Y	Y	Y		T
Jill	Y	Y	Y	Y	Y		T
Ken	Y	Y	Y	Y	Y		
Mia	Y	Y	Y	Y	Y	T	T
Olive	Y	Y	Y	Y	Y	T	T
Tengu	Y	Y	Y	Y	Y	T	T
Zach	Y	Y	Y	T	Y	T	T

Note. The code Y was given to students who made statements affirming the presence of the conception. The code N was given to students who made statements suggesting the conception was absent. The code T was given to students who made statements suggesting the conception was tentative, meaning uncertain, not fixed, or provisional. No code was given to students who did not make statements sufficient to award the other codes.

IC3a: Counterexamples to a claim can be described in terms of their properties (meets the condition but not the conclusion). In particular, this is a *pair* of properties simultaneously exhibited by each counterexample.
IC3b: The cases with the properties in IC3a comprise all of the counterexamples to the claim.
IC3c: Potential counterexamples may be hypothetical entities, especially if the claim is true.

IC3 and IC4 were more nuanced for some students than we anticipated, and we discovered subconceptions that appeared to be needed for some students to fully obtain an understanding of viable argumentation as eliminating counterexamples. IC3 and IC4 as originally written were still sufficient for describing the reasoning of students with robust understandings of the ECE framework, and yet some students did not possess all of the subconceptions. Bo, for example, expressed IC3a but did not exhibit IC3b in a manner that supported our viable argument vision.

The ICs above were uncovered in our current direct-argument context, and yet there is nothing in their descriptions that makes them specific to direct arguments. IC4b, on the other hand, was modified to include a conception that applies specifically to direct arguments:

IC4a: Descriptions of all possible counterexamples to a conditional claim can be useful in evaluating arguments.
IC4b: [Direct] Arguments for general claims are viable (or proof) if and only if counterexamples are eliminated by showing every case of the conditions also has the properties of the conclusion.

In addition, the following important, seemingly unproductive conceptions were also revealed in the analysis, some of which prevented the students from correctly analyzing the direct argument and may be considered obstacles to the use of the ECE framework:

- The belief that a viable direct argument may not eliminate all counterexamples and that there may be some sneaky counterexamples that a viable argument missed. This conception interferes with the use of the ECE framework and may reflect an overgeneralization of skepticism influenced by the skepticism lessons in the LLAMA intervention.
- A falsifiability condition for general mathematical claims: General mathematical claims can never really be proven true but can be proven false by counterexample. This belief may be influenced by student learning in science, where the criterion of falsifiability is essential to the scientific method. This conception reflects the importance to the ECE framework of the meta-understanding that it is possible to actually eliminate all counterexamples with a direct general argument.
- The understanding of how variables can represent all cases in the domain of a general claim. This is surely generally important for anyone assessing a general direct argument, not just for one applying the ECE framework.

5. Discussion

5.1. Significance and theoretical considerations

We found the ECE framework and the intermediate conceptions supporting it to be accessible and useful for most of the students in our study when assessing the viability of a direct argument. This supports our general perspective that identifying characteristics of potential counterexamples can help in direct argumentation. In some ways, the utility of the ECE framework in direct argument contexts is not surprising. Predicate logic theory for conditional claims of the form 'P implies Q' often involves truth tables with four bins for combinations of truth values for P and Q, based on the law of the excluded middle. If the tables are extended to quantified statements, *for all* x, *if* $P(x)$, *then* $Q(x)$,and the truth values are replaced with descriptions of the collections of objects $P(x)$, $\neg P(x)$, $Q(x)$ and $\neg Q(x)$, then the examples, nonexamples, and counterexamples tables used in our study are not too distant from notions and representations found in formal mathematical logic development. Similar to truth tables for $P \rightarrow Q$, our examples, nonexamples, and counterexamples tables focus students' attention on the bin that exclusively defines what it means for a statement to be false. In other words, our framework offers students an accessible way of acquiring one of the most important results from mathematical logic: that a conditional claim is true if and only if no objects in the counterexample bin exist.

While on the surface, a direct argument may not appear to address the existence of counterexamples at all, but the meta-theory that supports the validity of the mode of argumentation does. Consequently, having middle-grade students describe the properties of all possible counterexamples to a general claim and acknowledge how direct arguments eliminate the possibility of such examples may be a natural way to develop a meta-theory supporting the validity of direct argumentation that is akin to what they learn later in more advanced mathematics courses. Thus, our findings contribute to the field of viable argumentation and proof and proving by illustrating ways that students can access meta-level knowledge about what makes arguments viable, or proof, an area in which the existing research is limited.

Another theoretical consideration raised by the students' reasoning is that the ECE framework explicitly directed students' attention away from showing a *propositional* implication (P implies Q, or not Q implies not P) toward showing that *objects* with particular properties (i.e., counterexamples) could or could not exist. The subconceptions we discerned underscore some types of reasoning entailed by this redirection. First, it entailed the capacity to describe these objects in terms of their relevant properties (IC3a), in particular the properties that made an object part of the claim's domain but not its conclusion. Second, this description entailed an act of imagination (IC3c)—since the claim was true, these counterexamples were purely hypothetical, and in fact impossible, entities. The speculation associated with this approach involved further imagining of what else would be true about these hypothetical objects, until it became clear why they were impossible entities. This further underscores the breadth of mental activity involved when making mental models and leveraging those models to develop logical inferences (Johnson-Laird, 1983).

5.2. Implications for the use of the ECE framework in research and teaching

Despite the positive findings, we found several subconceptions or beliefs that served as supports or obstacles to the ECE framework; these subconceptions appeared to influence students' ability to leverage the ECE framework in a manner consistent with our vision. Our findings also highlight conceptions that serve as barriers to understanding and effectively using our framework to evaluate arguments. Future research might explore whether or not the subconceptions identified as barriers are truly barriers, or whether these subconceptions can provide pathways to more sophisticated understandings.

One prerequisite skill to using the ECE framework for direct arguments using variables is an understanding of the role of variables as placeholders for all cases in the domain of the claim and recognition that manipulations of these variables represent manipulations for all cases in the claim's domain. This is likely a prerequisite for anyone working with direct general arguments and is not unique to the ECE framework. This kind of issue about reading an argument and its representations and interpreting their meanings has been noted in numerous studies (see Stylianides et al., 2017, for a review). On the other hand, we note that seeing a generic example argument alongside the variable argument helped Zach make sense of the variable argument. Generic example arguments may thus provide occasions for students to apply the ECE framework to direct general arguments that do not use variables.

We view the ICs we identified as plausible elements of an understanding of the ECE framework, although we do not presume them to be exhaustive and do not assert that the conceptions appear in an ordered learning trajectory. In the ECE framework lessons, these ICs were taught explicitly, and students participated in activities for developing and applying them. Yet some of the students who understood IC1–IC3 and appeared to understand the argument presented to them expressed the idea that there are potentially other counterexamples to the claim that the argument did not eliminate (albeit perhaps "unimportant" counterexamples). This made the presence of IC4 tentative, perhaps absent. We are uncertain about the instructional implications for this finding and wonder if revisiting general claims with finite domains and exhaustive arguments might help. We wonder what these students might say in response to an argument in which all cases in the domain of the claim are tested. Would skepticism still be present? Would Zach still assert that the general claim is vulnerable to falsification by some unforeseen counterexample to be found by future researchers? If not, we wonder if understandings of exhaustive arguments and their ability to eliminate counterexamples can be leveraged to understand how general representations and valid transformations of the representations can do the same.

The ECE framework inherently builds upon the meta-understanding that all counterexamples can be ruled out, even by students with limited knowledge of context-level factors or of the role of variables and algebra in arguments. For students, this notion may be in conflict with notions taught in other classes. In science classes, student might gather data to support or develop their claims without necessarily arguing that there are no cases/examples to the contrary. Anecdotally, we have heard students and teachers say that you cannot prove a negative, and we have wondered how these statements might influence their conceptions of viable argument and proof in mathematics. Because some of the students in our study were concerned about whether or not all counterexamples to a general claim could be eliminated with an argument, we wonder if a meta-discussion addressing differences between knowledge and argumentation in mathematics and in other subjects might be worthwhile.

Numerous studies have documented that students at all levels are not sufficiently skeptical of arguments, particularly empirical arguments (see Stylianides & Stylianides, 2009, for a review). By contrast, we are not aware of studies that note students who are overly skeptical of mathematical arguments. We wonder if such skepticism is an unintended consequence of skepticism activities like those used in the LLAMA intervention. In all cases, these issues were found most often among students with lower mathematics achievement scores from the previous years, and these issues were not always consistently displayed by these students.

Our findings make this one of the few studies on a proof and proving intervention that appears to be effective, at least in a small case study environment. These findings offer opportunities to reflect on the amount and type of instruction needed to significantly improve knowledge and skill in proof, proving, and argumentation practices among students with diverse levels of academic achievement in mathematics, which is one of the reasons we chose to work with a group of students as opposed to an individual student as in Yopp (2017).

We also learned that while many proficient and advanced students acquired our practices and conceptions readily in response to our intervention, a few students did not. Most of these students had lower mathematics achievement scores. Even though these students were exposed to the same treatment, they expressed conceptions that interfered with their use of the ECE framework. In one case, a student's thinking was very different from what was presented in the intervention. This finding points to the complex intertwining of students' mathematical knowledge and skill and the various understandings needed for effective engagement in proof, proving, and argumentation. Further work is needed to explore the relationship between students' mathematics achievement status and students' learning of viable argumentation and proof.

5.3. Limitations and future work

This study was conducted in a small charter school in the United States that had a culture of argumentation in all subject areas. The classroom teacher was committed to this culture and to our project and its lessons as a way for students to acquire mathematical knowledge, practices, and skills. Viable argumentation, proof, and proving were regular features of the mathematics classroom of our sample students. In this way, the study's context may not be typical. However, the classroom where this study took place is a public school that could be viewed as typical in terms of the normal constraints in United States classrooms. The school faced similar pressures as other schools that teach Common Core State Standards for Mathematics and participate in standardized student achievement assessments.

With this in mind, future work could investigate how important the school and classroom culture of argumentation is to students' ability to understand and use the ECE framework. It is possible that students in schools that do not have a culture of argumentation as an important practice in all subjects would be less engaged in argumentation in mathematics. It would also be beneficial to augment the LLAMA intervention with ways to mediate the kinds of student reasoning issues, such as variable use to represent all possible cases and the logical implications found in algebraic transformation, that arose in this study and to examine the use of such an augmented ECE intervention.

Acknowledgements

This material is based upon work supported by the National Science Foundation Discovery Research K-12 Program under Grant No. DRL 1317034. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

Alibert, D., & Thomas, M. (2002). Research on mathematical proof. In D. Tall (Vol. Ed.), *Advanced mathematical thinking: Vol. 11*, (pp. 215–230). Dordrecht, Netherlands: Springer.

Balacheff, N. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematics learning. In E. von Glasersfeld (Ed.). *Radical constructivism in mathematics education* (pp. 89–110). Dordrecht, Netherlands: Kluwer.

Brown, S. A. (2014). On skepticism and its role in the development of proof in the classroom. *Educational Studies in Mathematics, 86*, 311–335. https://doi.org/10.1007/s10649-014-9544-4.

Cheng, P. W., & Holyoak, K. J. (1985). Pragmatic reasoning schemas. *Cognitive Psychology, 17*, 391–416. https://doi.org/10.1016/0010-0285(85)90014-3.

Durand-Guerrier, V. (2003). Which notion of implication is the right one? From logical considerations to a didactic perspective. *Educational Studies in Mathematics, 53*, 5–34. https://doi.org/10.1023/A:1024661004375.

Fan, L., Qi, C., Liu, X., Wang, Y., & Lin, M. (2017). Does a transformation approach improve students' ability in constructing auxiliary lines for solving geometric problems? An intervention-based study with two Chinese classrooms. *Educational Studies in Mathematics, 96*, 229–248. https://doi.org/10.1007/s10649-017-9772-5.

Fiallo, J., & Gutierrez, A. (2017). Analysis of the cognitive unity or rupture between conjecture and proof when learning to prove on a grade 10 trigonometry course. *Educational Studies in Mathematics, 96*, 145–167. https://doi.org/10.1007/s10649-017-9755-6.

Galbraith, P. L. (1981). Aspects of proving: A clinical investigation of process. *Educational Studies in Mathematics, 12*, 1–28. https://doi.org/10.1007/BF00.

Goos, M. (2014). Learning mathematics in a classroom community of inquiry. *Journal for Research in Mathematics Education, 35*, 258–291. https://doi.org/10.2307/30034810.

Harel, G. (2001). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell, & R. Zazkis (Eds.). *Learning and teaching number theory: Research in cognition and instruction* (pp. 184–212). Westport, CT: Greenwood Press.

Hub, A., & Dawkins, P. C. (2018). On the construction of set-based meanings for the truth of mathematical conditionals. *The Journal of Mathematical Behavior, 50*, 90–102. https://doi.org/10.1016/j.jmathb.2018.02.001.

Jahnke, H. N., & Wambach, R. (2013). Understanding what a proof is: A classroom-based approach. *ZDM—The International Journal of Mathematics Education, 45*, 469–482. https://doi.org/10.1007/s11858-013-0502-x.

Johnson-Laird, P. N. (1983). *Mental models: Towards a cognitive science of language, inference, and consciousness.* Cambridge, United Kingdom: Cambridge University Press.

Johnson-Laird, P. N. (2005). Mental models and thought. In K. J. Holyoak, & R. G. Morrison (Eds.). *The Cambridge handbook of thinking and reasoning* (pp. 185–208). Cambridge, United Kingdom: Cambridge University Press.

Johnson-Laird, P. N., & Wason, P. C. (1970). A theoretical analysis of insight into a reasoning task. *Cognitive Psychology, 1*, 134–148. https://doi.org/10.1016/0010-0285(70)90009-5.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge, United Kingdom: Cambridge University Press.

Larsen, S., & Zandieh, M. (2008). Proofs and refutations in the undergraduate mathematics classroom. *Educational Studies in Mathematics, 67*, 205–216. https://doi.org/10.1007/s10649-007-9106-0.

Lobato, J., Hohensee, C., Rhodehamel, B., & Diamond, J. (2012). Using student reasoning to inform the development of conceptual learning goals: The case of quadratic functions. *Mathematical Thinking and Learning, 14*, 85–119. https://doi.org/10.1080/10986065.2012.656362.

Mariotti, M. A. (2013). Introducing students to geometric theorems: How the teacher can exploit the semiotic potential of a DGS. *ZDM—The International Journal of Mathematics Education, 45*, 441–452. https://doi.org/10.1007/s11858-013-0495-5.

National Governors Association Center for Best Practices, & Council of Chief State School Officers (2010). *Common core state standards for mathematics.* Retrieved fromhttp://www.corestandards.org/wp-content/uploads/Math_Standards.pdf.

Popper, K. (1963). *Conjectures and refutations: The growth of scientific knowledge.* London, United Kingdom: Routledge.

Potari, D., Zachariades, T., & Zaslavsky, O. (2009). Mathematics teachers' reasoning for refuting students' invalid claims. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.). *Proceedings of the sixth congress of the European society for research in mathematics education* (pp. 281–290). Retrieved from: http://ife.ens-lyon.fr/publications/edition-electronique/cerme6/wg2-11-potari.pdf.

Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics, 83*, 323–340. https://doi.org/10.1007/s10649-012-9459-x.

Stylianides, A. J., & Al-Murani, T. (2010). Can a proof and a counterexample coexist? Students' conceptions about the relationship between proof and refutation. *Research in Mathematics Education, 12*, 21–36. https://doi.org/10.1080/14794800903569774.

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*, 314–352. Retrieved from https://www.nctm.org/publications/journal-for-research-in-mathematics-education/.

Stylianides, G. J., & Stylianides, A. J. (2014). The role of instructional engineering in reducing the uncertainties of ambitious teaching. *Cognition and Instruction, 32*, 374–415.

Stylianides, G. J., Stylianides, A. J., & Weber, K. (2017). Research on the teaching and learning of proof: Taking stock and moving forward. In J. Cai (Ed.). *Compendium for research in mathematics education* (pp. 237–266). Reston, VA: National Council of Teachers of Mathematics.

Weber, K. (2014). What is a proof? A linguistic answer to an educational question. *Paper Presented at 17th Annual Conference Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education*http://sigmaa.maa.org/rume/crume2014/Schedule/Papers.htm.

Weber, K., Maher, C., Powell, A., & Lee, H. S. (2008). Learning opportunities from group discussions: Warrants become the objects of debate. *Educational Studies in Mathematics, 68*, 247–261. https://doi.org/10.1007/s10649-008-9114-8.

Yopp, D. A. (2011). How some research mathematicians and statisticians use proof in undergraduate mathematics. *The Journal of Mathematical Behavior, 30*(2), 115–130. https://doi.org/10.1016/j.jmathb.2011.01.002.

Yopp, D. A. (2014). Undergraduates' use of examples in online discussions. *The Journal of Mathematical Behavior, 33*, 180–191. https://doi.org/10.1016/j.jmathb.2013.11.004.

Yopp, D. A. (2015). Prospective elementary teachers' claiming in responses to false generalizations. *The Journal of Mathematical Behavior, 39*, 79–99. https://doi.org/10.1016/j.jmathb.2015.06.003.

Yopp, D. A. (2017). Eliminating counterexamples: A Grade 8 student's learning trajectory for contrapositive proving. *The Journal of Mathematical Behavior, 45*, 150–166. https://doi.org/10.1016/j.jmathb.2017.01.003.

Yopp, D. A., & Ely, R. (2016). When does an argument use a generic example? *Educational Studies in Mathematics, 91*(1), 37–53. https://doi.org/10.1007/s10649-015-9633-z.

Zazkis, R., & Liljedahl, P. (2004). Understanding primes: The role of representation. *Journal for Research in Mathematics Education, 35*, 164–186. Retrieved from https://www.nctm.org/publications/journal-for-research-in-mathematics-education/.