A Geometric Path to the Concept of Function

Transformations using dynamic software can provide a unique perspective on a common topic.

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The concept of *function* spans elementary school through high school. When introducing the topic to elementary school and middle school students, we ask them to work with a variety of examples and representations of functions, since a formal definition of a function is rather abstract. These representations involve experiences in which students—

- use function machines to generate output from input,
- record input and output values in a table,
- substitute an input value in a formula to generate an output value, and
- graph input and output values using rectangular coordinates.

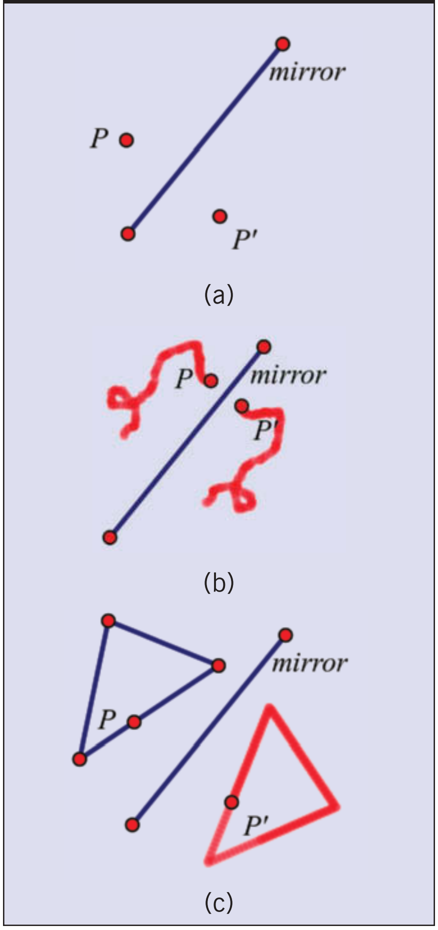
Our objective is to prepare students to deal with functions in an increasingly formal way by the time they begin studying algebra in late middle school or in high school. The most commonly used representations of functions are numeric in nature. Unfortunately, this emphasis on numeric experiences fails to acquaint students with the variety of mathematical relationships that can be represented as functions. As a result, it may contribute to common student misconceptions. Students may conclude that—

- every function turns an input number into an output number;
- every function can be expressed as an algebraic formula;
- a formula is the primary representation of a function, and that all other representations derive from it;
- every function is linear, particularly if linear functions are all they have seen (these same students may later come to believe that every function fits into a limited set of categories, such as linear, quadratic, polynomial, exponential, and so forth); and
- the ultimate test of a function requires graphing it in rectangular coordinates and applying the vertical line test.

A related problem is that commonly used representations of functions portray input and output in a discrete and static way, failing to emphasize the continuous behavioral characteristics of many functions. Students choose discrete input values and have few opportunities to manipulate variables continuously and directly. In a similar way, tables and graphs show multiple input and output pairs but fail to communicate a sense of continuous variation of the input and output variables. As a result, students may understand a variable acting as a placeholder only, for which they can substitute particular constants, without fully understanding how such variables can actually vary continuously. They see the function's behavior as a series of discrete snapshots rather than as a continuous movie.

Transformations *and* loci *are accessible geometric functions. Regrettably, we do not refer to them as* functions *and seldom encourage students to connect these two terms.*

Fig. 1 A geometric transformation is a function. The independent variable is the pre-image point *P*, and the dependent variable is the image point *P*′.



Many of these misconceptions can be addressed by giving students experience with representations that are nonnumeric and dynamic. Students can develop a deeper and more abstract view of functions by considering them in a dynamic geometry context (Hazzan and Goldenberg 1996). Dynagraphs (Goldenberg, Lewis, and O'Keefe 1992) are geometric representations of functions that lend themselves to making a connection between one-dimensional transformations and algebraic operations on the number line.

Other dynamic, nonnumeric representations are readily available. During the same middle school years when students explore representations of numeric functions, they already study functions that are geometric in nature. We give the most accessible geometric functions completely different names: *transformations* and *loci*. Regrettably, we do not refer to them as *functions* and seldom encourage students to connect these two terms.

In the case of transformations, we use different terminology: The input is a *pre-image* and the output is an *image*. In the case of loci, we do not give explicit names to the input and output.

A transformation, or a locus, takes a geometric object as input and produces a related geometric object as its output. These mathematical operations fit the definition of a function in a way that is readily accessible to middle school students and provide an important opportunity to expand and deepen their ideas about functions. When students explore such functions in a dynamic environment, they encounter the behavior of functions through the continuous variability of the input and output.

The remainder of this article discusses and provides examples of how exploring transformations and loci using dynamic geometry software, in this case The Geometer's Sketchpad®, can expand students' thinking about functions and encourage them to progress to a more sophisticated, abstract concept of these mathematical objects.

GEOMETRIC TRANSFORMATIONS AS FUNCTIONS

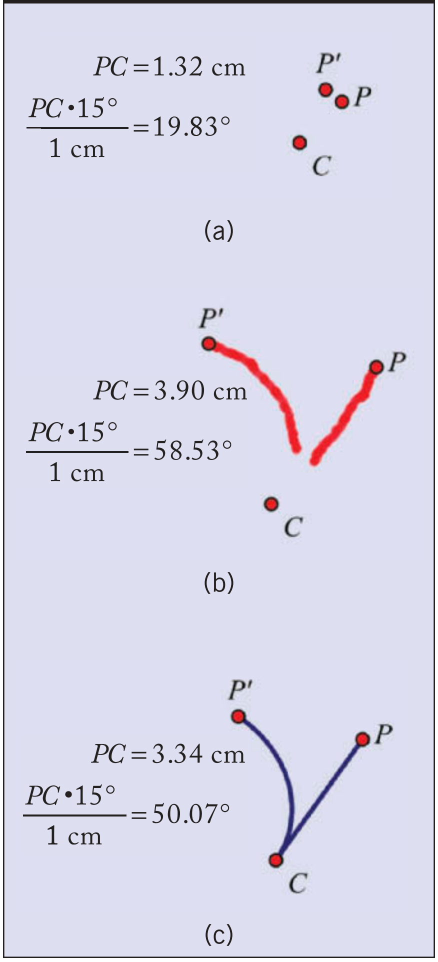
In an algebraic function, the input and output are both numbers. A geometric transformation can be defined similarly—as a function that takes a point as input and produces another point as output. For instance, a student might reflect a point across a mirror. **Figure 1a** shows the input point *P* and its reflection *P*′. As the student drags *P* to vary the input, the output—the position of *P*′—is observed.

In **figure 1b**, the student uses dragging to explore a portion of the function; the traces record corresponding portions of the domain and the range. By recording the locations of the input and output variables, the traces are equivalent to a table of values in the numeric realm, with a sense of continuous rather than discrete variation. In **figure 1c**, the student creates a triangle boundary and defines it as the domain by merging point *P* to it. The student then drags *P* around this domain to see *P*′ trace out the range.

Although students commonly explore reflections, slides, turns, shrinks, and stretches in middle school, they do not often look at them as functions that map an input point to an output point. An exploration such as this gives students the opportunity to consider how the output point depends on the input, to build up the transformed image point by point, and to consider the traces of the transformation's pre-image and image as the domain and range of a function. Students can use this same technique—transforming an input point to an output point—to explore slides, turns, stretches, and shrinks as functions.

Viewing transformations as functions opens up new and fascinating opportunities for students. In **figure 2a**,

Fig. 2 Custom transformations allow students to get a broader sense of function.



a student defines a transformation of input point *P* by measuring its distance from a given center point *C* and rotating *P* by an angle that depends on the measured distance. In this example, the student rotates *P* by 15 degrees for every 1 cm of distance, using the calculation shown in the figure to produce output point *P′*.

In **figure 2b**, the student drags and traces point *P* and observes the behavior of traced point *P′*. In **figure 2c**, the relationship between *P* and *P′* has been defined as a Sketchpad 5 "custom transformation," and this transformation has been applied to all the points on segment *PC*.

Because any point in the plane can be transformed in this way, students

9 Insights into Functions as a Result of These Activities

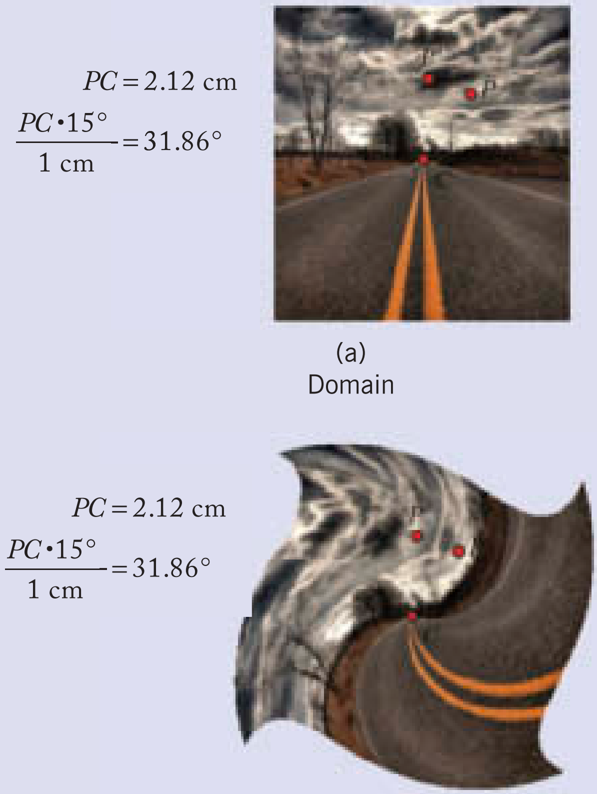
1. ***Functions need not be numeric.*** Students will quickly move beyond the simplistic and limited idea that a function takes a number as its input and uses a specific algorithm or formula to produce a second number as its output.
2. ***There is no one primary representation of function.*** Students should not be brainwashed by the idea that the "real" representation of a function is its equation and that other representations such as tables and graphs are merely windows on the representation as an equation.
3. ***Variables really vary*.** By dragging the input point of a transformation, locus, or dynagraph, students experience variation kinesthetically. The input becomes truly a variable.
4. ***Variables can vary continuously.*** Students can extend to the numeric realm their sense of continuous variation of dragged points. They can more easily imagine numeric quantities varying continuously, not just serving as placeholders for discrete values.
5. ***Functions can be viewed atomically or collectively.*** At first, students apply a function atomically, transforming a single input point to a single output point. Over time, they progress to applying the function collectively, transforming an entire set of points all at once.
6. ***In the collective view, functions map pre-image to image, domain to range.*** Students see a collection of points as a coherent set that they can use as input to a function, resulting in a similarly coherent set of output points. Working with these collections of points helps them understand the concepts of domain and range and see how a function maps domain to range.
7. ***Functions can be multidimensional.*** Like points on a line, numbers are one-dimensional. By experiencing geometric transformations as functions, students gain a sense that both input and output can be two-dimensional. Although they will not formally explore functions in two dimensions until much later, these early experiences provide them with valuable perspective.
8. ***Functions, transformations, loci, and mappings are related ways of expressing the same deep mathematical concept.*** These terms are used in different contexts and put more or less emphasis on the atomic or collective view. Students begin to integrate these terms into a more comprehensive whole.
9. ***Deep connections are found between algebra and geometry.*** Experiences with geometric functions build on students' early background with number lines, allowing them to experience variables and functions in the geometric realm. These experiences also prepare them for later encounters with functions that map pairs of coordinates in the plane to other pairs of coordinates.

Function Notation and Transformations

Zalman Usiskin and his colleagues at the University of Chicago School Mathematics Project (UCSMP) have advocated greater emphasis on transformations in K–12 geometry. The UCSMP high school textbook *Geometry* (Usiskin et al. 1997) uses function notation to describe some transformations. For instance, a student might write $\mathbf{r}_j(P)$ to indicate the reflection of point P across line j, and might write $\mathbf{r}_k(\mathbf{r}_j(P))$, or even $\mathbf{r}_k \circ \mathbf{r}_j(P)$, to indicate the reflection of point P first across line j and then across line k. By using function notation in the context of transformations, students have an opportunity to associate concrete images with the formalism and to make sense of the sometimes confounding right-to-left ordering of the symbols for the individual transformations.

Such experiences using function notation with geometric functions may be helpful in preparing students to use function notation with numeric variables. When students study composite functions, the mental image of $\mathbf{r}_k(\mathbf{r}_j(P))$ as a composite transformation that starts with point P, reflects across mirror j, and then reflects the result across mirror k can help students make similar sense of an algebraic composite function such as $g(f(x))$.

Fig. 3 A custom transformation provides a way to transform an entire picture.



(a)
Domain

(b)
Range

SETH BASISTA, FROM SKETCHPAD'S PICTURE GALLERY

can apply the custom transformation not only to the points that make up a segment but also to the points that make up an entire picture (see **fig. 3**). In this case, the domain is the rectangular pre-image (see **fig. 3a**) and the range is the swirled image (see **fig. 3b**). By applying the function to a picture, students can easily visualize how the function affects all the points in the domain defined by the pre-image.

This type of transformation is easily modified by changing the rule that produces the image. By changing the calculation, or by using transformations other than rotation, students can produce a wide variety of interesting functions. A student might create an angle parameter k, calculate the rotation ratio as $PC \cdot k/1$ cm, and observe the effect of different ratios on the behavior of the swirling function. Another student might investigate a function that dilates the input point by a ratio that depends on the distance from a center point. A third student might try a function that dilates the input point by a ratio that depends on the angle determined by the point, the dilation center, and a reference direction.

By engaging in such investigations, students develop a sense for how the relationship between input, P, and output, P', determines the resulting picture—how the two-dimensional transformation is built up by applying a function to many individual input points.

LOCI AS FUNCTIONS

Mathematically, a *locus* is a set of points that satisfy some particular mathematical condition. The condition is often expressed in terms of

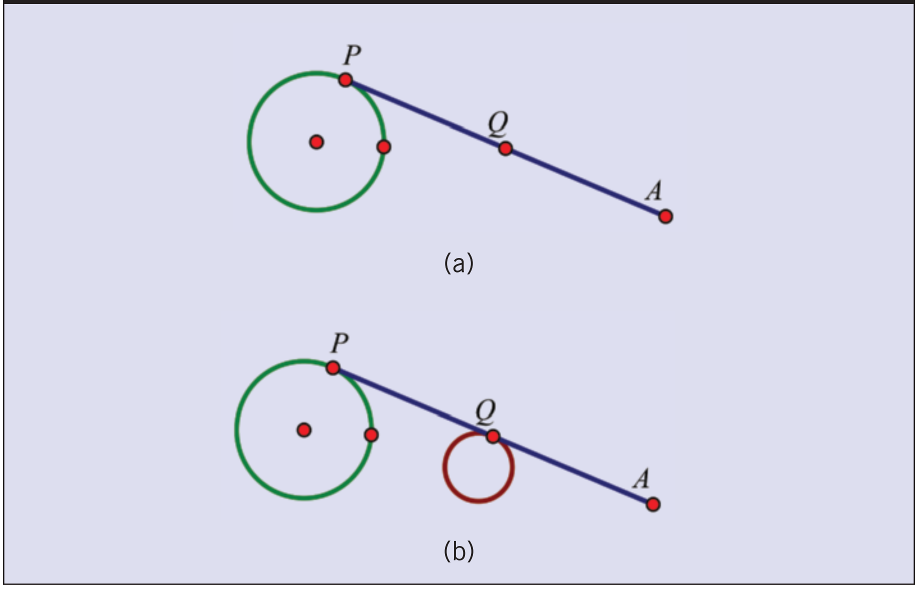


Fig. 4 As *P* moves around the circle, midpoint *Q* forms a new circle. The original circle is the domain of the function, and the new circle is the range.

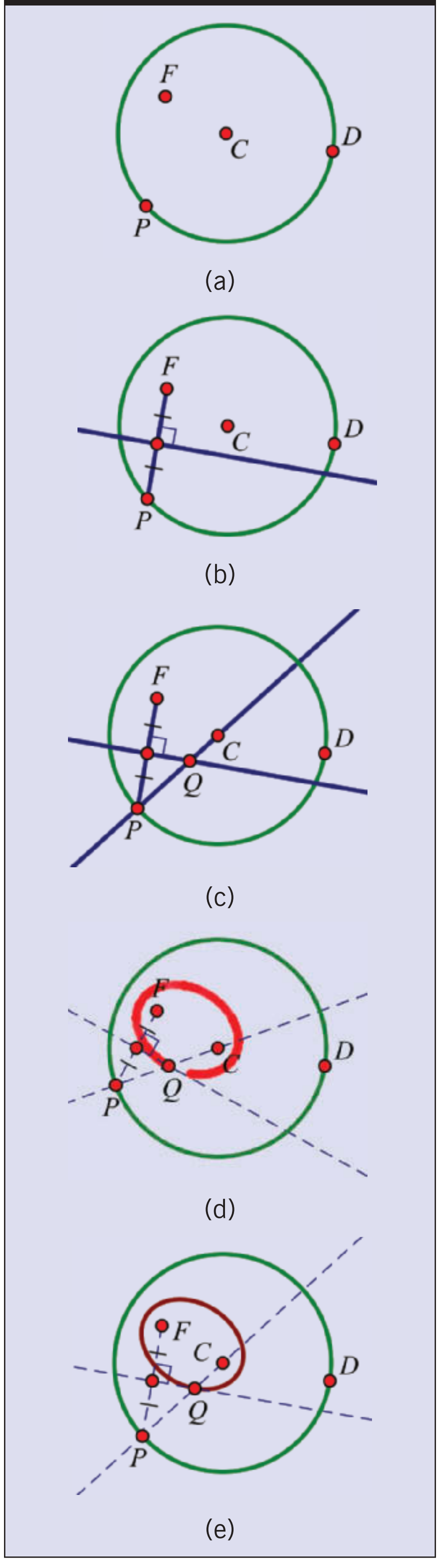


Fig. 5 With the introduction of independent variable *P*, the locus problem can be expressed in terms of a function: "Given point *P* on the circle, find point *Q* that's equally distant from point *F* and from the circle at point *P*."

some other point moving along a specific geometric path. Consider, for example, the triangle reflection example from **figure 1c**. We might ask, "What is the locus of *P*′ as *P* travels around the boundary of the triangle?" The common feature of both descriptions, as a transformation or as a locus, is that both are functions, with an independent variable, *P*, and a dependent variable, *P*′.

Consider the example in **figure 4** where segment *AP* has its endpoint *P* on a circle. **Figure 4b** shows the locus of midpoint *Q* of the segment as *P* moves around the circle. Although this locus is equivalent to shrinking the circle by 50 percent toward center point *A*, it is expressed differently, both in the student's conception and in the mechanism used to create it on the computer. But both mechanisms conform to the definition of a function: An independent variable, *P*, determines the position of a dependent variable, *Q*. Note how a locus differs from a trace. A *trace* is a static record of where the traced object has been; it does not change if the circle or point *A* is later moved. A *locus* is dynamic, responding immediately if either the circle or point *A* is dragged.

Another example starts with a circle defined by center point *C* and point *D* on the circle (see **fig. 5a**). Point *F* is a random point inside the circle. How can we find the locus of points equidistant from point *F* and the circle? To restate this problem in a form that involves a function, a student places an independent variable, point *P*, on the circle and changes the problem statement to, "Given point *P* on the circle, find point *Q* that's equally distant from point F and from the circle at point *P*."

To solve the restated problem, the student constructs the perpendicular bisector of segment *FP*. This perpendicular bisector is the set of points equally distant from *F* and *P* (see **fig. 5b**). One of the points along this line must be equidistant from *F* and the circle at *P*. Since the student knows that the distance from a point to a circle is measured along an extended radius, he or she then constructs the extended radius through point *P* (see **fig. 5c**). The student realizes that intersection *Q* is the point that satisfies the locus definition: It is equally distant from point F and from the circle at point *P*.

Many, but not all, locus problems can be similarly restated in terms of an input point and an output point. Problems that can be restated this way do not necessarily have a unique output point for each location of the input point, presenting an opportunity to discuss the difference between a function and a relation.

In **figure 5d** the student explores the function by tracing the dependent variable *Q* as he or she drags independent variable *P* around its domain. **Figure 5e** shows a constructed locus, which creates the elliptical range of *Q* corresponding to the circular domain of *P*.

Once the locus is constructed, the student can experiment with changes in the function definition by changing the domain (the circle) or the position of point *F* and observing the resulting change in the range. He or she can even make a surprising discovery when point *F* is dragged outside the circle. (Try it!)

An exploration like this encourages students to move from thinking about specific details (how the independent and dependent variables build up the locus one point at a time) to considering more general features (how the locus depends on the details of its definition). Investigating a locus in this way can help students make the transition from the atomic concept of mapping an independent point to a dependent point to the collective concept of mapping a more complex pre-image to image.

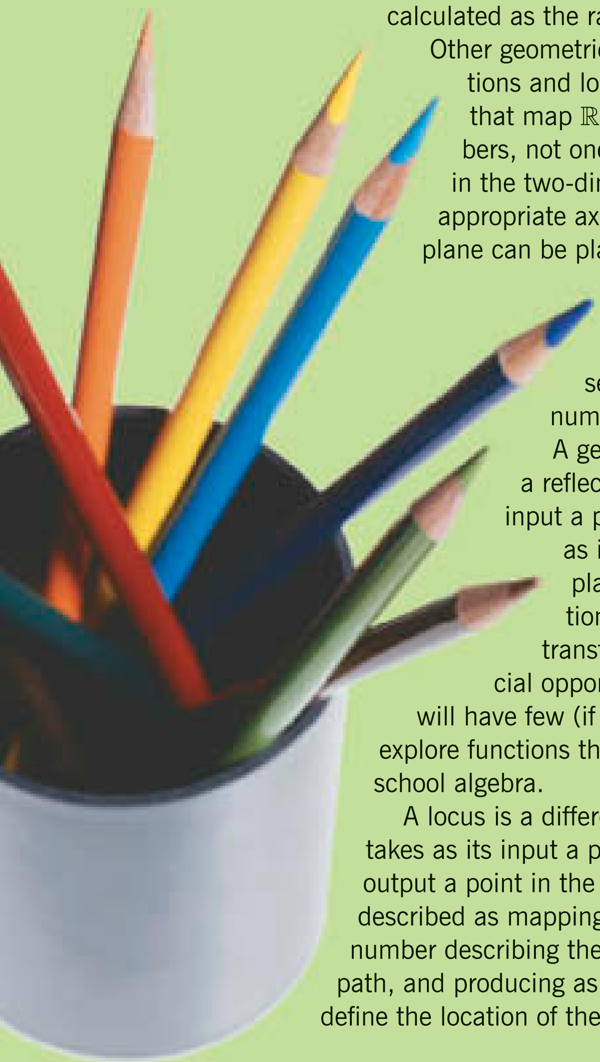
CONCLUSION

By giving students the opportunity to explore a variety of geometric representations of functions, we can expand their experiences and dislodge misconceptions that they might otherwise struggle to overcome later. In the process, they can gain valuable insight, grounded in concrete experience, into important mathematical concepts and principles.

Mapping $\mathbb{R}$ to $\mathbb{R}$, Mapping $\mathbb{R}^2$ to $\mathbb{R}^2$

Most functions that students will meet in their study of algebra map a single real number (the input variable) to another single real number (the output variable). Mathematicians describe such functions as mapping $\mathbb{R} \to \mathbb{R}$: mapping the real numbers to the real numbers. (Some functions have limited ranges, or restricted domains, so that the domain or range is actually a subset of $\mathbb{R}$ rather than all of $\mathbb{R}$.)

As math teachers, we often use the number line as a way to help students visualize $\mathbb{R}$. The number line expresses a deep connection between geometry and algebra; a mathematician might describe this in more abstract terms by saying that a number line puts the points on the line into a one-to-one correspondence with the elements of $\mathbb{R}$. For this reason, we can consider a function expressed in dynagraph form as equivalent to a function that maps $\mathbb{R} \to \mathbb{R}$; all we need to do is to turn the input and output axes into number lines. (We can do this by defining an origin point *A* and a unit point *B*; any other point *P* on the line can then be associated with the real number calculated as the ratio of directed distances *AP*/*AB*.)



Other geometric functions, such as transformations and loci, are not equivalent to functions that map $\mathbb{R} \to \mathbb{R}$, because it takes two numbers, not one, to define the position of a point in the two-dimensional plane. By defining appropriate axes and scales, the points in the plane can be placed into a one-to-one correspondence with ordered pairs of real numbers, such as (3.5, –2.91). Mathematicians designate the set of all ordered pairs of real numbers as $\mathbb{R}^2$.

A geometric transformation (such as a reflection across a mirror) takes as its input a point in the plane and generates as its output another point in the plane. We say that such a function is *mapping* $\mathbb{R}^2 \to \mathbb{R}^2$. Exploring transformations as functions is a special opportunity for students, because they will have few (if any) opportunities in algebra to explore functions that map $\mathbb{R}^2 \to \mathbb{R}^2$, even in high school algebra.

A locus is a different beast, because it generally takes as its input a point on a path and generates as output a point in the plane. Such a function can be described as mapping $\mathbb{R} \to \mathbb{R}^2$, taking as input a single number describing the position of the input point on its path, and producing as output a pair of numbers that define the location of the output point.

Of course, middle school students are not fully ready to develop a sophisticated understanding of functions and do not have a mature comprehension of the concepts listed here. But the concrete experiences we make available to them at this stage of their mathematical development can make a big difference in their early understanding of function and can prepare them to extend that early understanding in important ways. By allowing students to use dynamic mathematics software to investigate a variety of geometric functions, we plant ideas that will serve them well in their future mathematical development.

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Additional information about mapping real numbers and The Geometer's Sketchpad files are online at **www.nctm.org/mtms**.

A Geometric Path to the Concept of Function

Scott Steketee and Daniel Scher

The approach described in this article is supported by a number of Sketchpad activities, including worksheets, activity notes, and sketches. Several of the sketches can be downloaded from the article's summary page. All the worksheets, activity notes, and sketches are available from the Dynamic Number Project website at http://www.kcptech.com/dynamicnumber/welcome.html. (Click the "curriculum" link to find the Geometric Functions activities.)

These sketches give a flavor of some of the activities, but keep in mind that many of the activities are not based on prepared sketches but rather are designed so that students can create, manipulate, and investigate their own geometric functions.

ACTIVITY 1: IDENTIFY FUNCTIONS

Students use a prepared sketch to experiment with several examples and nonexamples of functions and formulate their own definition of function.

ACTIVITY 2: IDENTIFY FUNCTION FAMILIES

Students play several games in which they determine which one of four functions is different from the other three and in the process identify similarities and differences in the relative rate and direction in which variables change, and in the presence and locations of fixed points (locations where the independent and dependent points come together).

ACTIVITY 3: THE TRANSLATION FAMILY

Students create, manipulate, and investigate several different translations, with special attention to the behavior of these functions (the relative rate and direction of the variables and the presence and locations of fixed points). After investigating their own constructions, students undertake prepared challenges in which their job is to create a new function to match an existing translation. (The full activity requires the worksheet and activity notes; only the prepared challenges can be downloaded.)

ACTIVITY 4: THE REFLECTION FAMILY

Students create, manipulate, and investigate several different reflections, with special attention to the behavior of these functions (the relative rate and direction of the variables and the presence and locations of fixed points). After investigating their own constructions, students undertake prepared challenges in which their job is to create a new function to match an existing reflection. (The full activity requires the worksheet and activity notes; only the prepared challenges can be downloaded.)

ACTIVITY 5: THE ROTATION FAMILY

Students create, manipulate, and investigate several different rotations, with special attention to the behavior of these functions (the relative rate and direction of the variables and the presence and locations of fixed points). After investigating their own constructions, students undertake prepared challenges in which their job is to create a new function to match an existing rotation. (The full activity requires the worksheet and activity notes; only the prepared challenges can be downloaded.)

ACTIVITY 6: THE DILATION FAMILY

Students create, manipulate, and investigate several different dilations, with special attention to the behavior of these functions (the relative rate and direction of the variables and the presence and locations of fixed points). After investigating their own constructions, students undertake prepared challenges in which their job is to create a new function to match an existing dilation. (The full activity requires the worksheet and activity notes; only the prepared challenges can be downloaded.)

ACTIVITY 7: FUNCTION FAMILY DANCES

In this two-part activity, students dance several different functions from the four families they know (translation, reflection, rotation, and dilation). In the first part, groups of four students perform a dance for the class, with one student as the independent variable, another as the dependent variable, and the remaining two as the assistants who determine the particular member of the function family to be danced. In the second part, pairs of students use their mouse (or finger) to dance the role of the dependent variable as the independent variable dances along a restricted domain. (The full activity requires the worksheet and activity notes; only the sketch for the second part can be downloaded.)

ACTIVITY 8: FUNCTION DETECTIVE

Students review the characteristics of a city's four crime families (the Translation, Reflection, Rotation, and Dilation families) and use the evidence on each page of the sketch both to identify the family to which the criminal belongs and to determine the specific member of that family.

The eight activities described above are designed to introduce fundamental function concepts (emphasizing function behavior and families of functions) in a visual, dynamic way that is accessible to middle school students.

Additional activities address ways of combining functions (particularly composition of functions and inverses of functions) and the idea of a function as a mapping that can, in a single step, take a given set of input values and construct the entire set of corresponding output values (as shown in **figs. 2, 3,** and **4** in the article). Both are areas of study that are commonly regarded as advanced high school topics; however, the geometric approach makes them surprisingly accessible to middle school students.

All these activities are drafts, in various stages of field-testing and refinement. We will continue to polish these activities and support this geometric approach to functions with additional activities during the course of the Dynamic Number project. Check the Dynamic Number Project website for newer versions of these activities and for additional Geometric Functions activities as they become available.

All of the prepared sketches for these activities require Sketchpad 5. You can explore these sketches using the free preview version of Sketchpad 5 available at http://www.keypress.com/x24795.xml.

To help us improve the Geometric Functions activities, please send your suggestions and comments to Scott Steketee at stek@kcptech.com.

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