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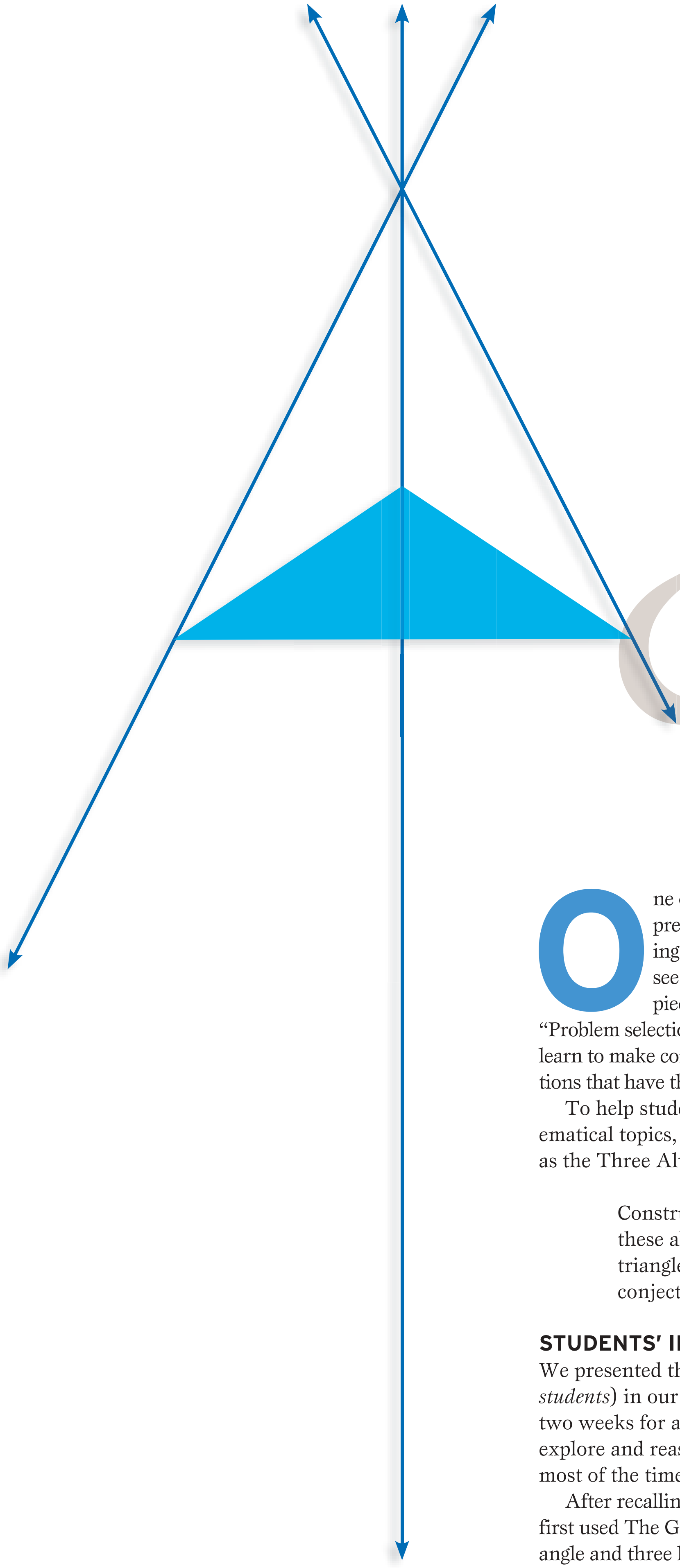


Proof & Mathe

One of the most rewarding accomplishments of working with preservice secondary school mathematics teachers is helping them develop conceptually connected knowledge and see mathematics as an integrated whole rather than isolated pieces. The NCTM Connections Standard (2000) states: "Problem selection is especially important because students are unlikely to learn to make connections unless they are working on problems or situations that have the potential for suggesting such linkages" (p. 359).

To help students see and use the connections among various mathematical topics, we have paid close attention to selecting such problems as the Three Altitudes of a Triangle problem:

> Construct the three altitudes of a triangle. What is true about these altitudes? Does your finding hold for other kinds of triangles? Define the *orthocenter* of a triangle. Prove your conjecture.

STUDENTS' INVESTIGATION AND THEIR CONJECTURES

We presented the problem to preservice teachers (referred to here as *students*) in our Geometry for Teachers class. Three class sessions over two weeks for a total time of four hours were provided for students to explore and reason about the problem. In each session, students spent most of the time working in small groups of two or three.

After recalling the definition of an altitude of a triangle, the groups first used The Geometer's Sketchpad (GSP)® to construct an arbitrary triangle and three lines, each through a vertex perpendicular to the opposite

Multiple Approaches

matical Connections

Using technology to explore the Three Altitudes of a Triangle problem, students devise many proofs for their conjectures.

side. Their constructions led students to observe that the three lines containing the altitudes (hereafter, *altitude lines*) meet in a point. When students dragged any of the vertices of the triangle to change its shape and size, they saw that their finding always held (see **fig. 1**). They summarized as follows: "For any triangle, the three altitude lines intersect at a single point, which is called the orthocenter of the triangle."

The process of making this conjecture seemed straightforward. However, as always when conducting mathematical explorations, students knew that the next step would be a challenge—either prove the conjecture or give a counterexample. So they continued their work on this problem in groups and tried to come up with a proof. We teachers circulated around the room, observing each group's work to monitor progress. Only when students had difficulties did we intervene by asking a question or a series of questions to lead them to think more deeply or think from a different viewpoint.

PROOFS USING EUCLIDEAN GEOMETRY

Proof 1

Some students in the class who had taken the college geometry course suggested that we apply Ceva's theorem to this problem. The theorem states:

Three lines drawn from the vertices A, B, and C of $\triangle ABC$ meeting the opposite sides in points D, E, and F, respectively, are concurrent if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

One student presented her group's proof to the class (see **fig. 2**). This proof works only for acute triangles, but similar proofs, with minor adjustments, can be done for obtuse and right triangles. This task is left for readers, for this proof and all other proofs discussed here.

From this group's proof, we can see that applying Ceva's theorem makes the proof easy. However, Ceva's theorem is usually introduced in a college geometry course, and its proof is generally more complex than directly proving the concurrency of the three altitudes or altitude lines. As a result, the proof would not greatly help high school geometry students understand the properties of altitudes or of the orthocenter. So we encouraged students to use the mathematical concepts that they learned in high school to prove their conjecture.

Aiming at generating some insightful ideas for other proofs, we guided students to do

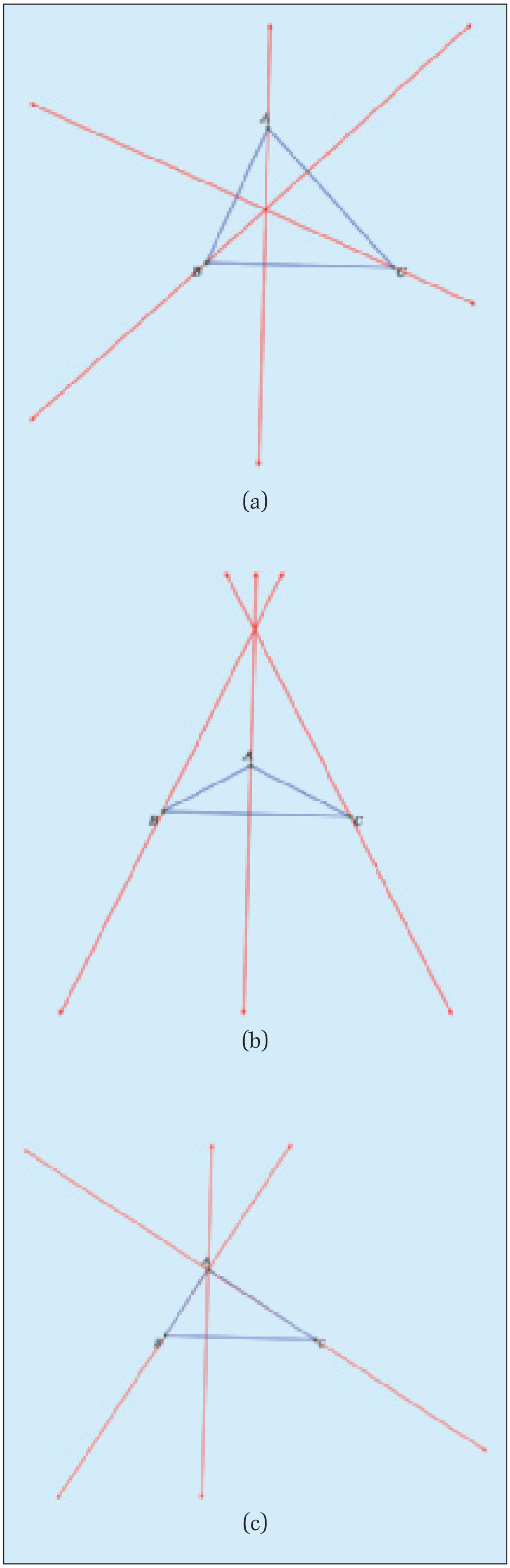


Fig. 1 The three altitude lines of a triangle intersect at a single point whether the triangle is acute, obtuse, or right.

further investigations with GSP. We suggested that students construct segments DE, EF, and FD in $\triangle ABC$ (see **fig. 3**). After the three segments were constructed to form $\triangle DEF$, the groups observed this new triangle and decided to measure the angles formed by the three altitudes and the three newly constructed segments. Using the measurements shown in **figure 3**, students discovered that the three altitudes of $\triangle ABC$ bisected the three interior angles of $\triangle DEF$—in other words, they were the three angle bisectors of $\triangle DEF$.

We further suggested that students hide segments DE, EF, and FD but construct three lines, each of which passed through a vertex of the original triangle parallel to its opposite side (see **fig. 4**), and that they investigate the resulting triangle formed ($\triangle LMN$). By measuring the angles and the lengths of the segments shown in **figure 4**, students discovered that each of the three altitudes was perpendicular to and bisected a side of $\triangle LMN$.

These investigations and findings stimulated students' thinking. Students had recently learned that the three perpendicular bisectors of any triangle are concurrent, as are the three angle bisectors of any triangle. Naturally, many students considered using these facts in their proofs.

Proof 2

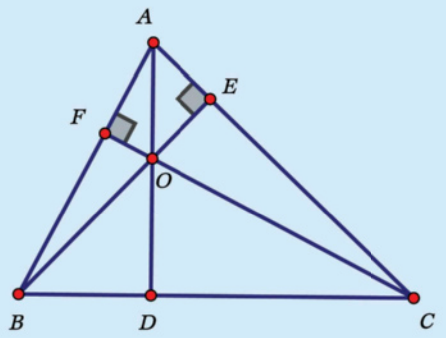
Choosing between the perpendicular and the angle bisectors, students found it easier to apply the theorem related to the three perpendicular bisectors and did so first. Students realized that to prove that an altitude line of $\triangle ABC$ is a perpendicular bisector of a side of $\triangle LMN$, they needed to prove that this line bisects a side of $\triangle LMN$. Through observation and group discussion of the construction shown in **figure 4**, students found that a proof could be accomplished by using quadrilaterals such as $ALBC$ and $NABC$, each of which could be demonstrated to be parallelograms. **Figure 4** shows a proof written by one group.

Proof 3

After making sure that students understood the proof shown in **figure 4**, we suggested that they try to prove that the three altitudes of $\triangle ABC$ are the three angle bisectors of $\triangle DEF$ (see **fig. 3**).

Students knew that they needed to prove that $\angle EDC \cong \angle FDB$ to show that $\angle EDA \cong \angle FDA$, a result indicating that altitude AD bisects $\angle EDF$. However, they experienced difficulties in seeing how to do so. To help them proceed, we first asked them the following leading questions:

- "Can you construct the circumcircle of $\triangle ABD$?"
- "What do you notice about the circle?"
- "How could you explain your finding?"



Given: $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the three altitudes of $\triangle ABC$.

Prove: $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ intersect at a single point.

Proof: Since $\triangle AFC$ and $\triangle AEB$ share $\angle FAE$, and each has a right angle, they are similar to each other by the Angle-Angle similarity property. So $\dfrac{AF}{AE} = \dfrac{AC}{AB}$ (Corresponding sides of similar triangles are in proportion). Similarly,

$$\triangle BDA \sim \triangle BFC \rightarrow \frac{BD}{BF} = \frac{AB}{CB};$$

$$\triangle CEB \sim \triangle CDA \rightarrow \frac{CE}{CD} = \frac{CB}{CA}.$$

By multiplying these three proportions, we get

$$\frac{AF}{AE} \bullet \frac{BD}{BF} \bullet \frac{CE}{CD} = \frac{AC}{AB} \bullet \frac{AB}{CB} \bullet \frac{CB}{CA} = 1.$$

That is, $\dfrac{AF}{FB} \bullet \dfrac{BD}{DC} \bullet \dfrac{CE}{EA} = 1$. Therefore, the three altitudes intersect at a point by Ceva's theorem.

Fig. 2 One group's proof applied Ceva's theorem.

Students constructed the circle that circumscribes $\triangle ABD$, although some needed a little help. They all noticed that point E appeared to be on the circle (see **fig. 5**). Further explorations led students to see quite a few "four points on a circle," or cyclic quadrilaterals, in the figure. Naturally, they thought of a way of using the theorems related to cyclic quadrilaterals to complete their proof.

As a class, we then reviewed the cyclic quadrilateral theorems and their converses to prepare students to apply these theorems:

- *Theorem 1:* If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.
- *Theorem 2:* If two angles inscribed in a circle intercept the same arc, then they are congruent.

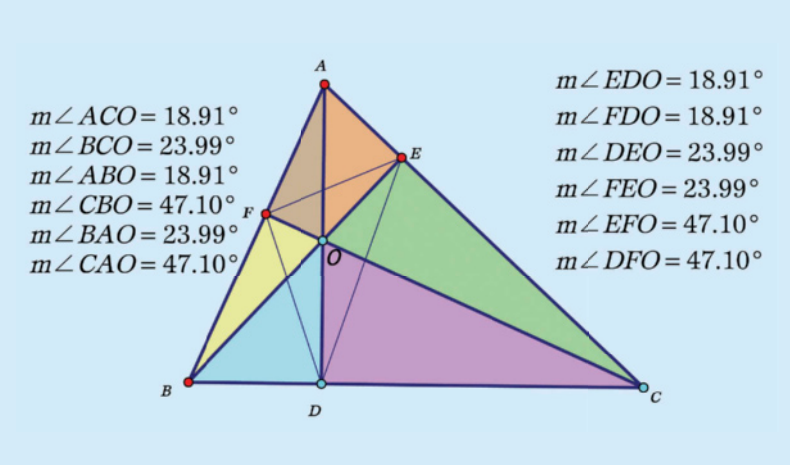
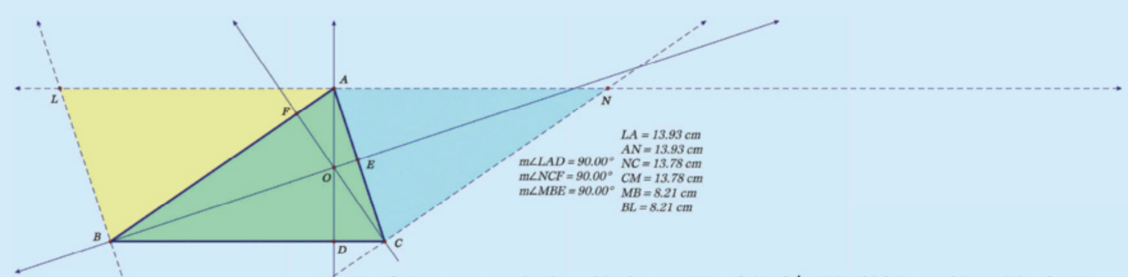


Fig. 3 The three altitudes of $\triangle ABC$ are the three angle bisectors of $\triangle DEF$.



Proof: Let's construct the three altitudes AD, BE, and CF of ΔABC, which is an arbitrary triangle. We want to prove that the three altitudes intersect at one point. To do so, let's construct $\overleftrightarrow{MN}$, $\overleftrightarrow{LN}$, and $\overleftrightarrow{LM}$ through the three vertices of ΔABC and parallel to $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, respectively, forming a large triangle, ΔLMN. In quadrilateral $ALBC$, we have $\overline{LA} \parallel \overline{BC}$ and $\overline{LB} \parallel \overline{AC}$ by construction. Then this quadrilateral is a parallelogram by definition. Therefore, $\overline{LA} \cong \overline{BC}$ because opposite sides of a parallelogram are congruent. Using similar reasoning, we can prove that quadrilateral $NABC$ is also a parallelogram, and so $\overline{AN} \cong \overline{BC}$. By transitivity, $\overline{LA} \cong \overline{AN}$. Since $\overline{LN} \parallel \overline{BC}$ and $\overleftrightarrow{AD} \perp \overline{BC}$, $\overleftrightarrow{AD} \perp \overline{LN}$. Therefore, $\overleftrightarrow{AD}$ is a perpendicular bisector of side $\overline{LN}$ of ΔLMN. In a similar fashion, we can prove that $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are the other two perpendicular bisectors in ΔLMN. By the finding we proved recently in class, the three perpendicular bisectors of any triangle intersect at a single point. $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, and $\overleftrightarrow{CF}$ are concurrent. This means that the three altitudes of ΔABC intersect at a point.

Fig. 4 The three altitudes of $\triangle ABC$ are perpendicular bisectors of the sides of $\triangle LMN$.

- *Theorem 3:* If opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle.
- *Theorem 4:* If two angles intercepting the same segment from its same side are congruent, then they can be inscribed in a circle, with the segment as a chord of the circle.

For complete proofs of these theorems, see Jiang and Pagnucco (2002).

Using these theorems, one group found a way to prove that $\angle EDC \cong \angle FDB$. Their reasoning follows:

> A, B, D, E [in **fig. 5**] are on a circle since both $\angle AEB$ and $\angle ADB$ are right angles and hence congruent, and both angles intercept the same segment AB from its same side (by theorem 4). Hence, $m\angle BAE + m\angle BDE = 180°$ (by theorem 1). But $m\angle EDC + m\angle BDE = 180°$ (a straight angle), and so $\angle EDC \cong \angle BAE$. Similarly (considering cyclic quadrilateral $ACDF$), we can prove that $\angle FDB \cong \angle BAE$. So $\angle EDC \cong \angle FDB$ by transitivity, i.e., altitude AD bisects $\angle FDE$. In a similar fashion, we can prove that the other two altitudes of $\triangle ABC$ respectively bisect the other two interior angles of $\triangle DEF$. Therefore, the three altitudes of $\triangle ABC$ are the three angle bisectors of $\triangle DEF$. Since the three angle bisectors of any triangle intersect at [a single] point, the three altitudes of $\triangle ABC$ are concurrent.

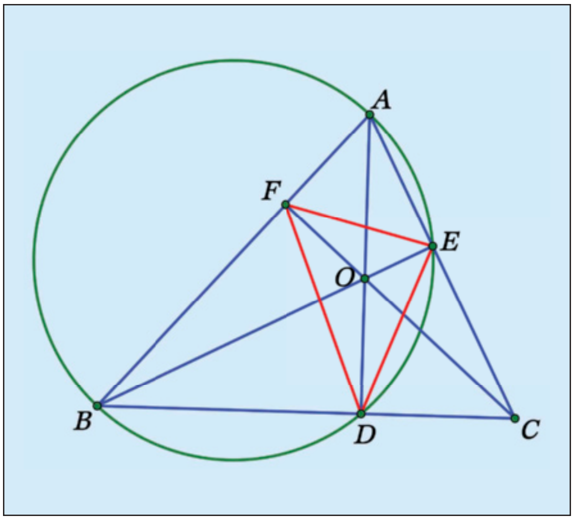
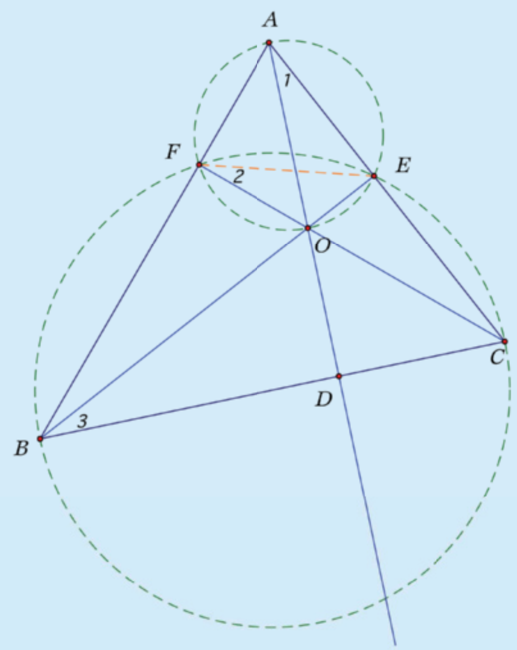


Fig. 5 *ABDE* is a cyclic quadrilateral.

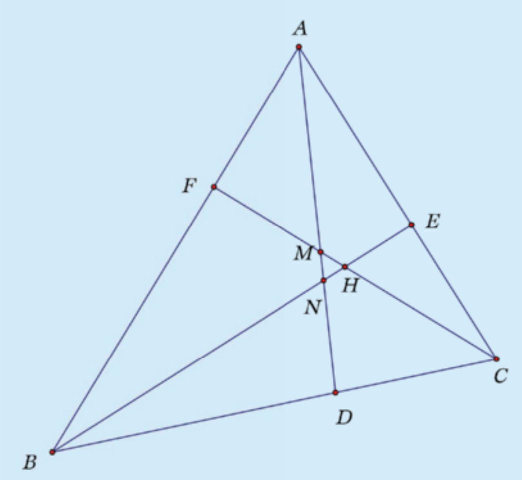
We asked this group of students to explain their reasoning as clearly as possible to the class and encouraged the other students to ask questions until they fully understood the proof.

Proofs 2 and 3 and the GSP investigations that facilitated insights into these proofs helped students see that mathematical ideas are interconnected: The altitude lines of a triangle are also the perpendicular bisectors, or angle bisectors, of a related triangle. Inspired by these new findings and proofs, students became curious and were eager to generate more proofs on their own. This was a wonderful teaching moment to engage students in further exploration and brainstorming and thus help them "develop an increased capacity to link mathematical ideas and a deeper understanding of how more than one approach to the same problem can lead to equivalent results …" (NCTM 2000, p. 354).



Proof: In ΔABC, construct altitudes BE and CF. Since the two altitudes are not parallel, they must intersect. Let O be their intersection point. Construct ray AO, which intersects $\overline{BC}$ at D. We need to prove $\overline{AD}$ is the third altitude of ΔABC, or, in other words, $\overline{AD} \perp \overline{BC}$. Quadrilateral $AFOE$ is a cyclic quadrilateral because a pair of opposite angles ($\angle AFO$ and $\angle AEO$) are both right angles and therefore supplementary (Theorem 3). Points B, C, E, F are on a circle since both $\angle BFC$ and $\angle BEC$ are right angles and hence congruent, and both angles intercept the same segment BC from its same side (Theorem 4). $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$ because each pair of the angles intercepts a same arc (Theorem 2). $\angle 1 \cong \angle 3$ by transitivity. In ΔADC and ΔBEC, there are two pairs of corresponding angles ($\angle 1$ & $\angle 3$, and $\angle ACD$ & $\angle BCE$) that are congruent or the same angle. By the triangle angle-sum theorem, the third pair of corresponding angles ($\angle ADC$ and $\angle BEC$) is congruent as well. Since $\angle BEC$ is a right angle, $\angle ADC$ must be a right angle. This means that $\overline{AD} \perp \overline{BC}$. Therefore, the three altitudes of ΔABC are concurrent (at point O).

Fig. 6 This is another proof involving cyclic quadrilateral theorems.



Proof: In ΔABC, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are the three altitudes. $\overline{AD}$ and $\overline{CF}$ intersect at M. $\overline{AD}$ and $\overline{BE}$ intersect at N. $\overline{BE}$ and $\overline{CF}$ intersect at H. In order to prove the three altitudes are concurrent, we need to show that M, N, and H are actually the same point. Since ΔAFM and ΔADB share $\angle FAM$ and each has a right angle, ΔAFM is similar to ΔADB by the Angle-Angle similarity property. Hence, $\frac{AF}{AM} = \frac{AD}{AB}$ as the corresponding sides of similar triangles are in proportion. So $AM = \frac{AF \bullet AB}{AD}$. In a similar fashion, we can prove that ΔAEN is similar to ΔADC and $\frac{AE}{AN} = \frac{AD}{AC}$. So $AN = \frac{AE \bullet AC}{AD}$. But ΔABE is similar to ΔACF because they share $\angle FAE$ and each has a right angle (by AA similarity property), and so $\frac{AE}{AF} = \frac{AB}{AC}$, which gives that $AF \bullet AB = AE \bullet AC$. Therefore, $AM = AN$, i.e., M and N are one point, which means that M is also on $\overline{BE}$, and hence is the intersection point of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$. (This also implies that M and H are one point, since $\overline{BE}$ and $\overline{CF}$ can only have one point of intersection.) Therefore, M, N, and H are one point, and all three altitudes of ΔABC intersect at a single point.

Fig. 7 This proof uses triangle similarity.

Proof 4

Small-group explorations and conversations continued. Several groups decided to use the cyclic quadrilateral related theorems that they had just reviewed in class to develop a new proof. With coaching and guidance from the teachers, they developed the proof shown in **figure 6**.

Proof 5

Prompted by the similar triangle concept used in the proof that applied Ceva's theorem, some groups decided to take advantage of the proportional relationship among corresponding sides of similar triangles. With only minor prodding from the teachers, they constructed the proof shown in **figure 7**.

A PROOF USING TRIGONOMETRY

While working through different approaches to a proof using Euclidean geometry, one group of students noticed that this problem involved many right triangles (e.g., $\triangle BCF$ and $\triangle AOE$ in **fig. 6**). They reasoned that this was a good opportunity to use right-triangle trigonometry to create a proof. As they did in the proof using cyclic quadrilaterals, students constructed two altitudes, BE and CF, in $\triangle ABC$. After labeling the intersection point of $\overline{BE}$ and $\overline{CF}$ as O, they constructed ray AO, which intersected $\overline{BD}$ at D (see **fig. 8**). Their goal was to prove, using trigonometry, that $\overline{AD} \perp \overline{BC}$. Their reasoning follows:

> $\angle ABE$ and $\angle ACF$ [in **fig. 8**] are congruent because each is complementary to $\angle FAE$. Let's use γ to represent each angle. In right triangle ABE, $\tan\gamma = AE/BE$. In right triangle OCE, $\tan\gamma = OE/CE$. So

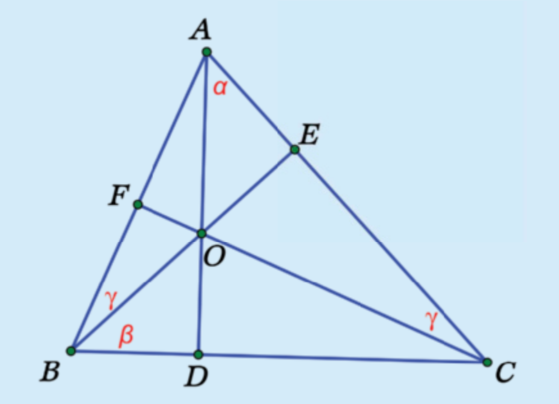


Fig. 8 The altitudes create right triangles and a chance to use trigonometry.

$$\frac{AE}{BE} = \frac{OE}{CE},$$

implying that

$$\frac{OE}{AE} = \frac{CE}{BE}.$$

In right triangle AOE, $\tan\alpha = OE/AE$. In right triangle BCE, $\tan\beta = CE/BE$. Therefore, $\tan\alpha = \tan\beta$. So $\alpha \cong \beta$. (This relationship holds because both α and β are acute angles.) Applying the triangle angle-sum theorem to $\triangle BCE$ and $\triangle ACD$, we have $\angle ADC \cong \angle BEC$ (a right angle), which means that $\angle ADC$ is also a right angle and that $\overline{AD} \perp \overline{BC}$. Therefore, $\overline{AD}$ is the third altitude of $\triangle ABC$, and the three altitudes intersect at point O.

This proof using trigonometry can be done a little differently by using similar triangles because all the proportions involved are the numerical relationships of the corresponding sides of related similar triangles. However, the advantages of using trigonometry are that it shows the application of trigonometry in geometric proofs and also makes finding those numerical relationships more convenient. Hence, students were able to see the connections between different branches of mathematics.

A PROOF USING COORDINATE GEOMETRY

In a homework assignment, we challenged students to find a coordinate geometry proof. Without any hint from us, several students were able to produce the proof shown in **figure 9**.

At a later class session, we asked students to compare the coordinate geometry method with the other methods used in the previous proofs. They commented that although they had enjoyed the previous proofs, they liked this method better: "It not only showed the connection between algebra and geometry, but additionally was more straightforward, making an otherwise difficult task much easier."

A PROOF USING VECTOR ALGEBRA

One student created a proof using vector algebra, as shown in **figure 10**. When this method was presented to the class at a later session, students were impressed by its beauty and compactness. They felt that vector algebra is easier to apply than geometry or trigonometry and requires knowledge of fewer rules, and they expressed interest in using vector algebra whenever appropriate in their future problem solving.

I have found that the three altitudes of the triangle are concurrent. This holds for all shapes of triangles.

Proof of the finding:

ΔABC is an arbitrary triangle with altitudes AD, BE, and CF. Assume altitude AD is located directly on the y-axis, and $\overline{BC}$ is on the x-axis. Let the coordinates of vertices A, B, and C be, respectively, $(0, a)$, $(b, 0)$, and $(c, 0)$ with $a > 0$, $c > 0$, and $c > b$.

We can find the slope of $\overleftrightarrow{BE} = \frac{c}{a}$ because the slope of $\overline{AC}$ is $\frac{a}{-c}$ and $\overleftrightarrow{BE} \perp \overline{AC}$. In a similar fashion, we can find the slope of $\overline{CF} = \frac{b}{a}$. Thus, the equation of $\overleftrightarrow{BE}$ is $y = \frac{c}{a}(x - b)$ and the equation of $\overleftrightarrow{CF}$ is $y = \frac{b}{a}(x - c)$.

To find the intersection point of $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$, $\frac{c}{a}(x - b) = y = \frac{b}{a}(x - c)$, which gives,

$$c(x - b) = b(x - c)$$
$$cx - cb = bx - bc$$
$$cx = bx$$
$$(c - b)x = 0$$
$$c \neq b \text{ (we assumed } c > b),$$

Thus,

$$x = 0.$$

Therefore, the two altitudes intersect on the y-axis, which is the third altitude.
So the three altitudes of the triangle are concurrent.

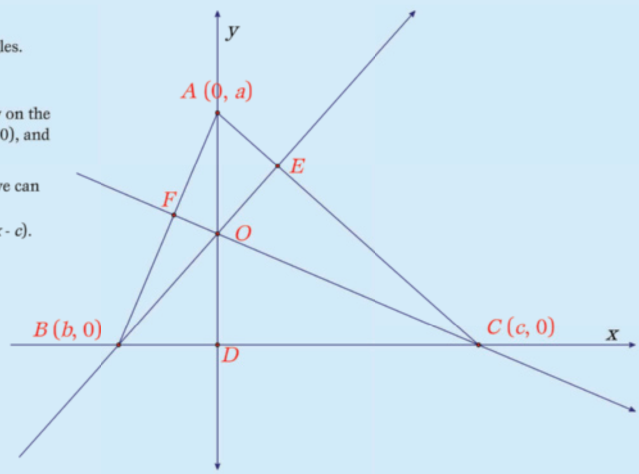


Fig. 9 Coordinate geometry can be used for an elegant proof.

Finding: The three altitudes of any triangle are concurrent.

Proof: In the graph, $\overline{BE}$ and $\overline{CF}$ are two altitudes of ΔABC, an arbitrary triangle.

Let O be the intersection point of $\overline{BE}$ and $\overline{CF}$. Construct $\overrightarrow{AO}$ intersecting $\overline{BC}$ at D. We want to prove that $\overline{AD} \perp \overline{BC}$, so $\overline{AD}$ is the third altitude, and the three altitudes are concurrent at point O.

Let vector$(OA) = \mathbf{a}$, vector$(OB) = \mathbf{b}$, and vector$(OC) = \mathbf{c}$.

Because vector(OB) · vector$(AC) = 0$, $\mathbf{b} \cdot (\mathbf{c} - \mathbf{a}) = 0 \rightarrow \mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{a} = 0$ (1)

Because vector(OC) · vector$(BA) = 0$, $\mathbf{c} \cdot (\mathbf{a} - \mathbf{b}) = 0 \rightarrow \mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = 0$ (2)

(1) + (2): $\mathbf{c} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{a} = 0 \rightarrow \mathbf{a} \cdot (\mathbf{c} - \mathbf{b}) = 0 \rightarrow$ vector(OA) · vector$(BC) = 0$

$$\therefore \overline{OA} \perp \overline{BC} \text{ and so } \overline{AD} \perp \overline{BC}.$$

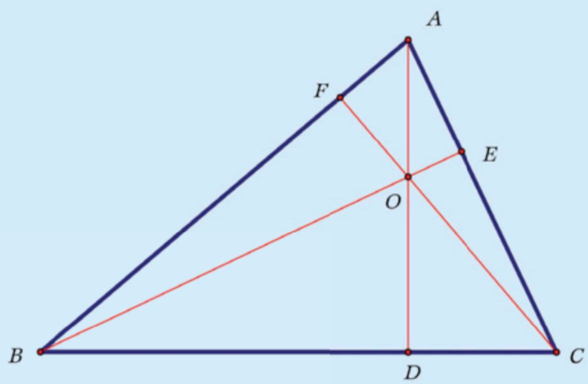


Fig. 10 This proof uses vector algebra.

CONCLUSION

After all the proof approaches were presented and discussed in class, students were excited about what they had learned and felt ownership of the proofs. They commented that they not only understood how to prove that the three altitude lines of any triangle are concurrent but also saw connections between this problem situation and various mathematical topics. In addition, their explorations of multiple approaches to proofs led beyond proof as verification to more of illumination and systematization in understandable yet deep ways (de Villiers 1999); expanded their repertoire of problem-solving strategies; and developed their confidence, interest, ability, and flexibility in solving various types of new problems. These benefits, in turn, will be passed on to their own students.

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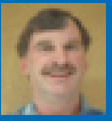

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