Exploratory Study of Informal Strategies for Equal Sharing Problems of Students With Learning Disabilities

Jessica H. Hunt and Susan B. Empson
Learning Disability Quarterly published online 16 October 2014
DOI: 10.1177/0731948714551418

Article

Learning Disability Quarterly
1–13

DOI: 10.1177/0731948714551418
ldq.sagepub.com

Exploratory Study of Informal Strategies for Equal Sharing Problems of Students With Learning Disabilities

Jessica H. Hunt, PhD[1] and Susan B. Empson, PhD[1]

Abstract

Little to no information exists explaining the nature of conceptual gaps in understanding fractions for students with learning disabilities (LD); such information is vital to practitioners seeking to develop instruction or interventions. Many researchers argue such knowledge can be revealed through student's problem-solving strategies. Despite qualitative differences in thinking and representation use in students with LD that may exist, existing frameworks of student's strategies for solving fraction problems are not inclusive of students with LD. This exploratory study extends existing literature by documenting the strategies students with LD use when solving fraction problems. Clinical interviews were conducted with 10 students across the third, fourth, and fifth grades (N = 10). Results indicate students with LD used similar strategies as previously reported in research involving non-LD students, although the dominant strategy utilized was less advanced and the range of strategy use was relatively compact. Researchers suggest the nature of conceptual gaps students with LD display in their understanding of fractions originates from a malleable source. Implications for instruction and assessment are presented.

Keywords

learning disabilities, cognition, student thinking, mathematics, fractions

Fractions are one of the most relentless areas of difficulty in mathematics for all students (National Center for Educational Statistics, 2009). Such difficulties affect students from the early elementary years through high school, where an incomplete understanding of fraction concepts interacts with students' ability to solve problems, apply computational procedures, and engage in algebra (National Mathematics Advisory Panel, 2008). Fractions seem especially difficult for students with learning disabilities (LD; Cawley & Miller, 1989). When asked to place fractions in order from least to greatest on two separate assessments, middle school students with LD answered only 47% and 1% of questions correctly, compared with 85% and 60% by students without LD (Mazzocco & Devlin, 2008). Researchers report the performance gaps suggest a lack of conceptual understanding. Hecht, Vagi, and Torgesen (2007) documented similar gaps in the elementary years, arguing that fourth- and fifth-grade students with LD begin their study of fractions with a diminished conceptual understanding compared with their peers. The consistent and pervasive difficulties students with LD experience with fraction concepts are alarming, as it is possible that diminished conceptual understanding has a cumulative effect on students' ability to learn more complex mathematics (Hecht & Vagi, 2010; Siegler et al., 2012; Vukovic, 2012). Yet, little to no information exists explaining the nature (or origins) of conceptual gaps for students with LD in understanding fractions; such information is vital to practitioners seeking to develop instructions or interventions.

In the following paragraphs, we first review descriptions of gaps and their possible sources in fraction conceptual knowledge for students with LD found in current literature, highlighting the dearth of information. Then, we explain how students' problem-solving strategies provide a window into students' conceptions of fractions and the nature of possible gaps, and introduce equal sharing problems as a context in which to study students' current conceptions. Next, we present a synopsis of the literature outlining strategies and thinking of children without LD when engaged in solving equal sharing problems. Finally, we introduce the current study and research questions.

[1]The University of Texas at Austin, USA

Corresponding Author:
Jessica H. Hunt, The University of Texas at Austin, Curriculum and Instruction, 1 University Station, Austin, TX 78712-0379, USA.
Email: jhhunt@austin.utexas.edu

LD, Fractions, and Gaps in Conceptual Knowledge: Possible Descriptions and Sources

Researchers disagree on explanations for the gaps in conceptual understanding of fractions for students labeled as LD. We describe two competing explanations found in the literature. One explanation suggests that these conceptual gaps stem from qualitatively different mathematical thinking. Lewis (2010) identified two misunderstandings of fraction concepts among adults with LD. Namely, "taking" (e.g., understanding shaded areas within fraction representations as "taken away" instead of as a part of a whole) and "halving" (e.g., understanding a part such as $\frac{1}{2}$ that is the result of an action as the action itself, sometimes evidenced as interpreting a partition line as a fraction), were atypical understandings evidenced by participants (relative to adults without LD). These misunderstandings seemed resistant to instructional intervention. Yet, because they were documented in adult learners, it is unclear whether children with LD would display similar misunderstandings. It is also unclear whether such conceptions result from years of instruction in which these learners had little support to develop conceptual understanding or from initial qualitative differences in how students with LD think about fractions. Similarly, Van Garderen and Montague (2003) also claimed that the mathematical thinking of children diagnosed with LD was qualitatively different from other children's thinking, in the realm of whole-number problem solving.

A second explanation posits cognitive factors that are not specific to mathematical thinking as the source of the mathematics difficulties experienced by students with LD. A wealth of research has explored the possibility that broad cognitive factors, such as working memory or processing speed, preclude students from using developmentally appropriate thinking to solve problems and thus develop conceptual knowledge of fractions (Davis et al., 2009; Siegler, 2007). However, this body of research does not converge on any cognitive factors specific to conceptual understanding of mathematics for students with LD (Mazzocco & Kover, 2007). In fact, in virtually every study conducted on LD and mathematics to date, researchers document individual strengths and weaknesses across varying content domains, which include solving word problems in mathematics and basic computation (Compton, Fuchs, Fuchs, Lambert, & Hamlett, 2012). In short, there is little to no consensus about which or how many, if any, cognitive factors associated with LD in mathematics would be pertinent in research that aims to define the nature of conceptual differences in understanding fractions in students with LD.

Defining the Nature of Conceptual Gaps Through Mathematical Activity

The qualitative diversity of LD and the ambiguity of cognitive factors suggest that research aiming to document the nature of conceptual gaps students with LD evidence in fractions might begin with an analysis of students' mathematical activity as they engage in solving problems (Simon et al., 2010; Steffe, 2002; Tzur, Johnson, McClintock, & Risley, 2012). Such analysis would provide a starting point to aid educational practitioners and researchers to "understand the components and developmental progression of students' [conceptual understanding] to guide reliable assessments and interventions for children with LD" (Vukovic, 2012, p. 300). The documentation begins with research that (a) uncovers students' current conceptions of fractions as they engage in solving fraction problems and (b) documents related factors that may be influencing students' initial conceptions of fractions and their engagement in problem solving (Tzur et al., 2012).

Students' Conceptions Revealed in Problem Solving

Although there are many ways to define and document conceptual understanding, we used students' problem-solving strategies as evidence of students' understanding (Charles & Nason, 2000; Empson, Junk, Dominguez, & Turner, 2005; Siegler, 2005; Steffe & Olive, 2009), because strategies can "serve as a behavioral marker for children's internal concepts, beliefs, and understandings of problems [and content]" (Fazio & Siegler, 2013, p. 55). Specifically, students' strategies for equal sharing problems—equally sharing some number of same-sized objects among some number of people, where the result is a fractional quantity—elicit students' conceptions of fractions (Empson & Levi, 2011; Streefland, 1993). Students' strategies for equal sharing problems and conceptions of fractions advance in tandem. Several researchers have documented frameworks outlining students' strategies, representations, and language use for equal sharing problems (Charles & Nason, 2000; Empson & Levi, 2011; Streefland, 1993). We synthesized this rather extensive body of empirical research on students without LD into the following framework, which we used in the current study.

Framework for students' problem solving in equal sharing problems. Students' initial conceptions of fractions can be elicited by equal sharing tasks in which the number of objects to be shared is greater than the number of people sharing so that each person gets, essentially, a mixed-number amount (e.g., four people sharing nine candy canes). This type of task extends children's understanding of similar situations in which each person gets a whole-number amount (e.g., four people sharing eight candy canes). In the most basic strategies for equal sharing problems, students may not exhaust the quantity to be shared or they may create unequal shares (a *No-Coordination* strategy). Here, students are attending to either the necessity to use up all the items to be shared or the need to share everything, but not both at the same time.

Students may also use strategies based on notions of counting to partition a whole into some number of "parts." For example, to share nine candy canes between four people, a student may give two candy canes to each person and have a leftover.

To share the leftover, a student may repeatedly halve until they have enough parts to distribute to each person or use knowledge of simple fractions to cut the leftover into four parts. Yet, when asked to quantify each person's share, the student may say "three" or "two and one half," because the parts are not seen in relationship to a whole and may not, in the student's mind, be differentiated from the whole (Steffe & Olive, 2009). These strategies are called *Non-Anticipatory* because, while the student is now attending to both the need to exhaust all items to be shared and make the shares equal, the partitioning and subsequent naming of the fractional quantity produced is not associated with a relation between the number of sharers and the amount being shared (Empson et al., 2005). In other words, when initially conceiving of the situation, the student does not anticipate that partitioning the amount to be shared is related to the number of people sharing. Instead, the student uses trial and error or repeated halving to create parts to distribute equally.

As students continue to work with equal sharing situations and their conceptual notions of fractions evolve, they begin to use *Emergent Anticipatory* strategies, in which they anticipate prior to the activity of partitioning each item the relationship between the amount to be shared and the number of sharers (Empson et al., 2005; Steffe & Olive, 2009). To share nine candy canes between four people, students use the previously described relation between sharers and the amount being shared combined with flexible understanding of composite units (e.g., four is four units of one or one unit of four) to mentally plan the partitioning of the leftover, one or two items at a time.

In activity, the student begins to differentiate, or disembed, $\frac{1}{4}$ of a whole from the whole itself (Steffe & Olive, 2009). The student sees the partitioned item as parts of a whole and may quantify one person's share as "two and one fourth" with words or symbols.

Students' conceptions of unit fractions such as $\frac{1}{2}$ and $\frac{1}{4}$ deepen as their attention shifts from equality and size among the parts to equality and size of the parts relative to the whole. They begin to see a whole, for instance, as four fourths and one whole all at once and the relationship between the whole and the parts as reversible (Olive & Steffe, 2002; Tzur, 1999). For instance, to share three French fries among four people, a student may represent each fry and split each one into four equal parts because there are four people and then use addition to describe one person's share as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$.

In a strategy reflecting a more sophisticated understanding, students imagine the partition of each fry into fourths without having to represent each fry and use multiplication to combine the unit fractions $\left(3\times\frac{1}{4}\right)$ for an equal share of $\frac{3}{4}$. The student understands $\frac{3}{4}$ as a composite unit comprised of $\frac{1}{4}+\frac{1}{4}+\frac{1}{4} = 3\times\frac{1}{4}$ (Empson et al., 2005; Steffe & Olive, 2009).

The emergence of a multiplicative conception of fractions is reflected in students' ability to think in a distributive manner about the equal sharing division $3 \div 4$ as $(1 \div 4) + (1 \div 4) + (1 \div 4)$ (Empson & Levi, 2011; Steffe & Olive, 2009). With time, this coordination becomes a mental recall of the activity of equally sharing three items among four groups—an *Anticipatory* strategy reflecting students' understanding of the relationship between a value of fraction (e.g., three fourths of a whole unit) and dividing the numerator by the denominator (e.g., 3 divided by 4).

Research Questions

Despite the richness of the literature in mathematics education concerning how children come to understand fraction concepts and their strategies for equal sharing, there is a dearth of similar literature concerning how students with LD might conceptualize such tasks, the nature of conceptual differences in the strategies students use to solve problems, and factors that may influence students as they solve problems. This exploratory case study extends existing literature by documenting what conceptual understandings students with LD exhibit as they work with equal sharing problems and the extent to which students' thinking appears to be consistent with or differ from the framework described above for students without LD. Our aim was to address the following research questions:

Research Question 1: What initial conceptions of fractions (i.e., employed strategies and representations) do children with LD evidence as they work with equal sharing problems?

Research Question 2: To what extent, if any, does the nature of students' thinking or representations used to solve equal sharing problems appear to differ from existing frameworks?

Method

Participants

Ten students across the third, fourth, and fifth grades (N = 10) participated in the study. We defined inclusion criteria for study participants as follows: (a) currently in the third, fourth, or fifth grade; (b) having a label of LD; and (c) having individualized education program (IEP) goals in mathematics. Purposeful sampling procedures were used to formally identify participants (Brantlinger, Jimenez, Klinger, Pugach, & Richardson, 2005). Namely, we relied on teacher characterization and performance on state mathematics testing to consider a broad range of performances within the inclusion parameters. We used this sampling procedure to maximize the possibility that a broad range of strategies would emerge during the course of the study. After interviewing 10 students, no new strategies emerged (i.e., saturation) and data collection concluded. We collected demographic information regarding characteristics of students relating to ethnicity, gender, LD label, and grade level from the district at the onset of data collection. A summarization of student demographic data by school is displayed in Table 1.

Setting

Eight of the interviews took place in small classrooms across three different elementary schools located in a large urban city in the southern United States. Two of the interviews took place in a small classroom at a university-tutoring clinic for elementary school children in a small rural town in the northwestern United States. Interview sessions generally lasted 1 hr; researchers used additional sessions as needed to complete problem tasks.

Problem Tasks

We designed a set of six problem tasks for use in the study based on the work of Empson and Levi (2011). For each of the tasks, the researcher presented mathematical problem situations to students based on a story context (e.g., 4 friends shared 14 soft tacos, so that each of them got the same amount to eat. How many tacos did each child eat if they finished all of the tacos?). The problem-solving tasks were designed so that students could use a variety of strategies and representations to reason about the mathematics and come to a solution.

Each of the six problem tasks was situated in equal sharing (e.g., two people share five items; four people share three items) situations, as employed strategies on equal sharing tasks were the main focus of the study. In each equal sharing problem, the number of sharers ranged from two to four and the number of objects shared ranged from 3 to 14. Problems were designed to elicit fractional values greater than 1 (i.e., number of items is greater than number of sharers) and less than 1 (i.e., number of items is less than number of sharers). Problems that resulted in values greater than 1 (e.g., 4 share 14) were asked first in an attempt to link to students' prior understandings of partitive division with no remainders (e.g., 4 share 12). Selected problems are listed in Table 2.

Study Design and Procedures

Researchers conducted a standardized clinical interview (Ginsburg, 1997) with each student individually in a small classroom equipped with large tables, manipulative materials (i.e., unifix cubes, paper rectangles that could be drawn on or torn), writing instruments, and paper. Each student was presented with a series of tasks; for each, the interviewer began with a problem designed to elicit a fractional value greater than 1. The student and the interviewer read each problem orally. A strategy for solving the problem was not presented. Instead, students were encouraged to solve each problem in a way that made sense to them—they could use the manipulative materials, paper and pencil, or no materials to aid them in reaching a solution. The interviewer pressed students to explain and justify each of their solutions in an attempt to understand their thinking processes. The interviewer repeated each student's answers/statements back to them to encourage student elaboration. When the student produced a representation, the researcher asked what the drawing or symbols represented. The researcher also took anecdotal notes during each interview conducted.

Interviews were designed to reveal as much as possible about each student's understanding and thus were dynamically adapted depending on students' responses. Students' strategies and fraction terminology all informed the interviewer in terms of which task to administer next. In general, task administration began with two problems where the result was greater than 1 (e.g., 2 share 5 yields a solution of $2\frac{1}{2}$). The next problem given usually involved a result greater than 1 but could produce a non-unit fraction result (e.g., 4 share 14 results in $3\frac{2}{4}$, or $3\frac{1}{2}$ depending on the strategy used). Next, problems with a result of less than 1 and a non-unit fraction answer were planned (e.g., 4 share 3 yields $\frac{3}{4}$; 5 share 2 yields $\frac{2}{5}$; 3 share 8 yields $\frac{3}{8}$).

In some instances, the order in which we presented tasks and the wording was varied to discourage rote approaches from one task to the next and/or to respond to student thinking (Ginsburg, 1997). For example, in cases where students were conceptualizing a couple of problems using a particular strategy, we might give a more difficult task (e.g., with a result less than 1) to see if they continued to utilize this strategy or used a different strategy. If the student employed a less advanced strategy, the interview may have returned to the easier problem. The interviewer also individualized the context of each problem situation to student preference.

Table 1. Characteristics of Students.

Characteristic	School 1 (%) n = 2	School 2 (%) n = 4	School 3 (%) n = 2	School 4 (%) n = 2
Age				
8–9		50		
10–11	50	50	100	
12–13	50			100
Gender				
Male		25		100
Female	100	75	100	
Ethnicity				
Caucasian	50	25		100
African American		25		
Hispanic	50	50	100	
Disability				
Math LD			100	
Reading LD	50	50		
Math and reading LD	50	50		100

Note. LD = learning disabilities.

Each student solved an appropriate number of problems such that trends in their thinking could be observed. That is, tasks were administered to students until it was evident that (a) no new strategies or insights into how students with LD solved fraction problems emerged or (b) students could no longer provide a solution to the problems on their own or with minimal prompting.

Coding and Analysis Procedure

To analyze the data, all interviews were audiotaped and transcribed verbatim; the audio files were destroyed. The transcriptions were then entered into a Microsoft Excel spreadsheet; corresponding written student work and anecdotal notes taken during the interviews were assembled to ensure triangulation of data. Data analysis was done on three levels; the first level employed a constant comparison method to delineate two indicators of a student's concept of fractions: their employed strategy to solve a problem that resulted in a fractional quantity and the representations used. First, researchers read through the full transcriptions of all interviews as a team. Next, the data were chunked into smaller, more meaningful parts (i.e., solutions to each problem posed). Then, researchers independently labeled each problem with a descriptive title (i.e., code) related to strategy type and the nature of employed representations. Codes were then compared using peer debriefing and collaborative work (Brantlinger et al., 2005). Inter-rater reliability $\left(\frac{\text{agreements}}{\text{agreements+disagreements}}\right)$ across strategy codes was 80%. All initial discrepancies were resolved through peer debriefing among the coders. There were four disagreements that were resolved as No-Coordination strategies and five disagreements that were resolved as Non-Anticipatory strategies. It is noteworthy that any initial disagreement was due to a student's strategy reflecting elements of two possible strategy codes (Ginsburg, 1997; Siegler, 2005, 2007).

We adapted and defined the strategy codes deductively from a priori categories (Leech & Onwuegbuzie, 2007) focused on the nature of employed strategies and supporting representations used to solve problems. A previously established delineation of children's strategies for solving equal sharing problems (Empson & Levi, 2011), described above, served as the framework for types of strategies and representations observed in the data. Throughout the constant comparative level of analysis, researchers compared each new problem solution and its code with previously coded data to ensure consistency (Leech & Onwuegbuzie, 2007). After all problem solutions were coded, the codes were grouped and a name was identified for each grouping.

The second level of data analysis used classical content analysis. We employed this analysis to discern how many times student used certain strategies across the interviews. This descriptive information about the data complemented the constant comparative analysis used earlier (Leech & Onwuegbuzie, 2007). Researchers used the strategy codes and grouping names for the analysis. For the name given to each grouping of strategy codes, researchers counted how many occurrences comprised each grouping (e.g., how many times a No-Coordination strategy was coded) in the data set. We then divided the totals by the total number of all coded strategies to obtain a percentage of time each grouping was used (e.g., No-Coordination strategies comprised 15% of all coded strategies).

Table 2. Six Interview Tasks.

Task	Example	Possible prompts
5 ÷ 2	Lidia and Jerome shared 5 soft tacos so that each of them got the same amount to eat. How many tacos did Lidia and Jerome each eat if they finished all of the tacos?	They want to share this taco, too (if the shares are not exhausted). What if they each want the same amount (if shares are uneven)?
14 ÷ 4	14 sticks of clay are shared among 4 children for a project. How much clay does each child receive?	What if they all want the same amount of clay (if shares are uneven)? What if they want to share all of the clay (if shares not exhausted)?
4 ÷ 3	3 friends share 4 large chocolate bars so that they all get the same amount. They eat all of the chocolate bars. How many cookies does each friend get?	What if we have to keep everything in the problem the way it is? What if we have to keep four candy bars and we have to keep three people?
3 ÷ 4	4 friends share 3 small pizzas. If they each want the same amount and share all of the pizzas, how much pizza does each friend get?	What if they want to eat all of the pizzas? Remember, they share the pizzas so that they each get the same amount.
2 ÷ 5	5 friends share 2 submarine sandwiches so that each friend gets the same amount. How much sandwich does each friend receive?	Remember, they each want the same amount of sandwich to eat. How might you split the first sandwich? The second?
3 ÷ 8	There are 3 bottles of water that 8 people want to share equally. How much of the water does each person receive in his or her cup?	They each drink the same amount of water so it has to be equal. How might you split the first bottle? The second? The third?

Finally, because prior research suggests the possibility that students with LD may evidence "atypical" conceptions of fractions (Lewis, 2010; Mazzocco & Devlin, 2008), we employed emerging coding within a constant comparison analysis in an attempt to capture instances of any atypical thinking/explanations noted in the literature review or factors that may have interfered with students' employed strategies in some manner. We found 20 such instances in the data. To code/name each instance, we first read through the full transcriptions of all interviews, examining each problem solution. Next, when we found an instance where problem solving seemed to be troublesome to a child within a solution, we noted the child's spoken and gestured actions within that solution, applying a fine-grain assessment approach to the child's way of solving the problem (Siegler, 2007). Then, each researcher gave the occurrence a name. Finally, researchers compared these names and resolved any disagreements. The initial agreement among the researchers on names was 15 out of the 20 instances (75%). Researchers resolved all disagreements.

Results

The aim of this exploratory case study was to document LD students' conceptual understanding of fractions as evidenced in their employed strategies for equal sharing problems. We found three broad categories of problem-solving strategies: (a) No-Coordination, (b) Non-Anticipatory, and (c) Emergent Anticipatory. We also found three categories of instances that interfered with problem solving: (a) rote use of strategies, (b) lack of ownership, and (c) impediments specific to a given task. Table 3 summarizes indicators of each strategy and associated interfering factors; Table 4 lists the tasks each student was given and their employed strategy.

No-Coordination Strategies

No-Coordination strategies were the most basic strategies employed by students in the study. Students who used No-Coordination strategies showed no evidence of an a priori understanding of the relationship between the amount being shared and number of sharers. These strategies generally involved one of two outcomes: (a) not exhausting the amount to be shared or (b) exhausting the amount to be shared by making unequal shares. Fractional quantities were not created. Students' No-Coordination strategies generally involved the recognition of the need to share a number of objects among a number of sharers. These strategies were only observed for problems in which the number of objects to share was greater than the number of sharers. Students used a basic direct modeling strategy that involved dealing objects to sharers in some fashion and then they either ignored the leftover amount or attempted to partition the objects and distribute the parts, but without creating equal parts or equal shares.

No-Coordination strategies comprised 15% of all employed strategies. Some students used concrete objects, such as cubes, to directly model problems. Other students

Table 3. Strategies Used in Equal Sharing Problems by Students With LD.

Strategy	Characteristics of strategy
No-Coordination	Student uses dealing strategy or partitioning to share quantities, but does not exhaust amount to be shared or gives out unequal whole number shares.
	Student does not create fractions or quantify each person's share.
Non-Anticipatory	Student uses skip counting, iteration of equal-sized groups equal to the number of sharers, halving, or trial/error and knowledge of simple fractions to conceptualize problem.
	Student creates fractions but may or may not name each person's share.
Emergent Anticipatory	Partitioning and naming related to the number of sharers.

Note. LD = learning disabilities.

used pictorial representations, such as circles or rectangles, to show their thinking. In all cases, no fraction terminology was evident in children's responses because the strategy did not result in the creation of fractional quantities.

A third-grade student displayed this strategy as she worked to solve an equal sharing problem involving five tacos and two sharers:

S: This is easy. It says "the same amount" so these are the key words. [Draws five groups of two tacos]. Ten tacos.
I: Ten tacos. How did you know it was ten tacos?
S: I saw five and two so I drew five equal groups of two.
I: You drew equal groups. OK. Tell me what the five groups represent.
S: I made five equal groups of tacos.
I: OK. How do your five groups of tacos relate to this problem?
S: [Looks at interviewer for a bit] Well, they. . . . I don't. . . .. My teacher didn't show me this yet.
I: Hmm. What might you do if you had to share those tacos? Can you picture them in front of you on a plate?
S: Oh! Like this . . . one for you, one for me. One for you, one for me [pauses and pushes leftover to the side].
I: [Watches]
S: [Looks at the interviewer for a bit] So they [one sharer] would get three and they [the other sharer] would get two tacos.
I: What if they both wanted the same amount?
S: There's not enough.

Non-Anticipatory Strategies

Unlike No-Coordination strategies, children's Non-Anticipatory strategies showed a nascent understanding of fractional quantities. The emergence of a rudimentary level of coordination between sharers and the amount being shared distinguished Non-Anticipatory strategies from No-Coordination strategies. In Non-Anticipatory strategies, children exhausted the amount to be shared, creating fractional quantities in activity. Similarly to No-Coordination strategies, students began by recognizing the need to share a number of objects among a number of sharers and using some form of direct modeling to distribute objects to sharers. However, in contrast to No-Coordination strategies, when students reached the point where they had a leftover number of objects that was less than the number of sharers, they used knowledge of common fractional quantities to partition the objects into smaller parts to distribute. In choosing a common fraction, such as halves or fourths, students did not take into account the number of sharers. With trial and error or repeated halving, they were able to exhaust the sharing material and may or may not have used correct fraction terminology to describe the final share.

In the data, 76% of the strategies employed to solve equal sharing problems were Non-Anticipatory, making this strategy the most prevalent among the children in our study. Students supported their Non-Anticipatory strategies by using tally marks, drawn figures, and paper shapes to represent the sharers and the amount to be shared in each problem. Fraction terminology was sometimes evident in children's responses at this level of understanding, although the fraction terminology used to name the shares was often "one-half," regardless of the actual fraction quantities created. Other times, children referred to each person's share as a number of pieces.

For example, a fourth-grade student used a Non-Anticipatory strategy to solve a task involving three people and four pizzas:

S: There are four pizzas [draws the three children and puts out four cubes for the pizzas]. These can be the pizzas and these can be the three children [Deals out one pizza to each child and sees one left. Writes "4 - 3 = 2"]. I think it's two because there are four pizzas and three children.
I: OK.
S: I don't know . . . I have to figure out another way to explain it.
I: OK.
S: OK so that's the three children. I wanted to give a pizza to each child because each one wanted to get as

Table 4. Tasks and Coded Strategies by Student.

Student	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6
1	5 ÷ 2 *Coder 1: NA* *Coder 2: NC* *Final code: NC*	4 ÷ 3; NA	14 ÷ 4 *Coder 1: EA* *Coder 2: NA* *Final code: NA*	3 ÷ 4; NC		
2	5 ÷ 2; NA	4 ÷ 3; NA	14 ÷ 4; NA	3 ÷ 4; NA	2 ÷ 5; NA	
3	5 ÷ 2; NA	4 ÷ 3; NA	3 ÷ 4; EA	2 ÷ 5; NA	3 ÷ 8; EA	
4	5 ÷ 2; NC	4 ÷ 3; NA	14 ÷ 4 *Coder 1: EA* *Coder 2: NA* *Final code: NA*	3 ÷ 4; NA	2 ÷ 5; NA	
5	5 ÷ 2 *Coder 1: NA* *Coder 2: NC* *Final code: NC*	4 ÷ 3; NA	14 ÷ 4 *Coder 1: EA* *Coder 2: NA* *Final code: NA*	3 ÷ 4; NA	2 ÷ 5; NA	
6	5 ÷ 2 *Coder 1: EA* *Coder 2: NA* *Final code: EA*	14 ÷ 4; NA	4 ÷ 3; NA	3 ÷ 4; NA	2 ÷ 5; NA	
7	5 ÷ 2 *Coder 1: NA* *Coder 2: NC* *Final code: NC*	14 ÷ 4 *Coder 1: EA* *Coder 2: NA* *Final code: NA*	4 ÷ 3; NA	3 ÷ 4; NC		
8	5 ÷ 2 *Coder 1: NA* *Coder 2: NC* *Final code: NC*	4 ÷ 3; NA	3 ÷ 4; NA	2 ÷ 5; NA		
9	5 ÷ 2; NA	4 ÷ 3; NA	3 ÷ 4; NA	2 ÷ 5; NA	3 ÷ 8; NA	
10	5 ÷ 2; NA	4 ÷ 3; NA	14 ÷ 4, NA	3 ÷ 4; NA	2 ÷ 5; EA	

Note. Italics depict initial disagreement in codes resolved through collaborative work. Initial and final codes are given. NC = No-Coordination; NA = Non-Anticipatory; EA = Emergent Anticipatory.

much as they could. So that's one, two, three [deals a cube out to each person].

I: OK.

S: And there's one left.

I: There's one left? What if they want to share that one too?

S: Well they could probably have it or someone else could probably have it.

I: What if they want to have it?

S: We could probably put it in the middle and they could split it up [motions over the cube]. I don't know how to split it.

I: Oh! Could I make a suggestion? Could you draw this last one . . . because you just said something about splitting it up and we can't physically split that cube up.

S: Sure this will be the little cube [draws a square] and we can split it [splits it in half, then draws a second square and erases]. I don't know.

I: Is it hard to split that square?

S: Yeah.

I: Could you draw that last cube another way?

S: Yeah [draws a circle and partitions it into halves] It's a pizza so I drew a circle but I can't . . . it's hard.

I: Oh. Pizzas aren't always round. I ate a rectangular pizza once. What if the pizza looked like a rectangle?

S: [draws rectangle]. Oh! I know right there . . . and right there [easily partitions rectangle into three pieces]. So there are the children [draws a stick figure above each third].

I: So that's each person's share of that one pizza. And you gave each one whole pizza earlier. So then how much pizza does each person get?

S: They get one . . . one and one half.

I: One and one half?

S: Yes because there's getting one whole pizza and this represents one whole pizza and this is the split up one

. . . one-third, one-third, one-third. So one and . . . one and one half.

Another example of a Non-Anticipatory strategy to solve a problem involving 14 tacos and 4 sharers can be seen in the transcript of a fifth-grade student below. Like the previous student, this student began by not wanting to create fractions by splitting two of the tacos; this was overcome by a simple prompt from the interviewer. In contrast to the previous student, this student referred to each person's share as a number of pieces as opposed to using fraction terminology in their final quantification:

S: [Begins to attempt to multiply 4 × 14].
I: Can you tell me why you are doing that?
S: It helps me to multiply 4 times the 14 to equal the answer . . . what this is.
I: OK. [Grabs bag of color tiles]. Can you use these to show me your thinking?
S: OK. [Counts out 14 color tiles] I get one you get one . . . [proceeds to divide the color tiles into two groups].
I: OK. But the problem says we have four people sharing this time.
S: Oh! [Redistributes the tiles and gives two of the groups five and two of the groups four].
I: Is that fair that they have different amounts?
S: No . . . oh. OK. So they . . . those are left [points to two].
I: What could we do with those?
S: We don't give those out.
I: What if each person wanted to get all that they could get?
S: You'd have to split those two up . . . into . . . fourths.
I: OK. So, how much would each person get if we did that.
S: [points to each group as she talks] Well, they get one and they get one and they get one and they get one . . .
I: OK. And how about all together? What would each person get?
S: They each get five.
I: Five?
S: Because we started with three first and then we each got two more pieces.
I: When we shared these last two . . . what were these pieces called?
S: One fourth, one-fourth.
I: OK. So when we cut up these into fourths, are they the same size as this whole one?
S: No they are different. [Points to wholes] These three are different and these are the same size. When they split this one got smaller.
I: OK. So can we still count 1, 2, 3, 4, 5 . . . if they are different?
S: Yes, they are five.
I: OK.

Emergent Anticipatory Strategies

Emergent Anticipatory strategies reflected evidence of an understanding that, to produce equal shares that exhausted the amount to be shared, fractional quantities related to the number of people sharing must be used. Emergent Anticipatory strategies involved children splitting each item to be shared into a number of parts equal to the number of people sharing. Similarly to previous strategies, the child begins with the recognition that a number of objects need to be shared among a number of sharers. However, at the point where there are more sharers than there are objects to be shared, the child decides to partition each object (or sometimes, a small group of objects such as a pair) into a number of parts that is equal to the number of sharers; this action is repeated until there are no more objects to share. Generally, to quantify the share, the child names each part as a unit fraction within the boundaries of the whole (e.g., one item partitioned into six parts produces "sixths") and counts the number of fractional parts each sharer receives. To quantify the fractional parts, some children use addition and others indicate the total share by marking their strategy in some way.

In our data, only 9% of the strategies employed to solve equal sharing problems fell into this category. Emergent Anticipatory strategies were used to solve problems that resulted in answers both greater than and less than 1. Students directly modeled the sharers and the amount to be shared in each problem. Fraction terminology was evident in children's responses at this level of understanding, although several students continued to use the term "one-half," regardless of the actual fraction quantities created. An example of a fourth grader's strategy to equally share three bottles of soda among four children follows:

S: They could like get a cup or something [draws four cups]. Four cups.
I: Four cups . . .
S: And we could do it like this [splits each bottle into fourths but calls them halves] halfway, halfway, halfway, and halfway. So they all get a drink from each bottle.
I: How much does each person get?
S: One.
I: One? One what?
S: One half of each.

Factors That Interfered With Students' Problem Solving

We documented a number of factors that appeared to interfere with students' problem solving and the use of their conceptual understanding. These are factors that were not necessarily cognitive in origin. Some seemed be artifacts of instruction; that is, students had appropriated orientations to problem solving that interfered in some way with their productive engagement in the tasks we presented to them. Others could be specific to the child and have other origins. Some problem solutions we coded contained more than one interfering factor (e.g., sometimes, students evidenced rote strategies and specific impediments to enacting the task, such as representational barriers).

Rote strategies are strategies that children most likely learned in instruction that are applied based on superficial features of a problem. In the example of a No-Coordination strategy listed above, the student's initial attempt to solve the equal sharing problem was based on the use of the keywords, "the same amount." Keyword strategies are an artifact of instruction and have been shown to be ineffective problem-solving supports (Garofalo & Lester, 1985). In this case, it led the student to conceptualize the problem as equal groups multiplication. This initial orientation to the problem impeded the student's spontaneous use of a valid strategy. However in this instance as well as all others, the interviewer was successful in redirecting the student's attention toward conceptualizing the entire situation rather focusing only on a single word or phrase. Eight instances of the use of rote strategies were found, all of which occurred during No-Coordination and Non-Anticipatory strategies. This factor appeared among children of all grade levels.

A second interfering factor was coded as *lack of ownership* of mathematics knowledge. Students who displayed this factor during problem deferred to the teacher for explicit direction on how to solve a problem. We found seven instances of this interfering factor and it was exclusive to Non-Anticipatory strategies. A fifth-grade student displayed this factor before she used a Non-Anticipatory strategy to solve a problem involving 14 enchiladas and four sharers:

S: [Makes piles of four cubes to a total of 12 . . . add another pile for a total of 16. Thinks for a while.]
I: What are you thinking?
S: This one's harder because [long pause] nothing times 4 equals 14.
I: That's kind of like the other one . . . times two . . . like nothing times two equals five. What if you imagine sharing them . . . all 14 enchiladas here. How would we share those?
S: I don't know . . . still hard.
I: What about drawing a picture or using tiles again. . . . I can't imagine 14 in my head. Could you use these or draw a picture maybe that could help you figure it out.
S: [Puts down four tiles. Then deals out 14 tiles one by one to each person, gets unequal piles]. It is not going to be equal.
I: I wonder if you could make them equal. They all want the same amount.
S: I don't know. You know, though. Can't you show me?
I: I'm interested in how you are thinking about it.
S: Well, I guess. . . . I was thinking [puts face in hands] you could try splitting them in half or something?
I: Split them in half. Why would we split them in half?
S: Is that right?
I: [listens]
S: Let's see if it's fair to the other groups . . . they had one . . . one . . . then they would each have to have one there. It's fair.
I: So you can make them equal. So how much would each person get?
S: Three and a half.
I: Three and a half? That sounds like a good amount.

Sometimes students had difficulty moving forward in their solutions because a representation was difficult to manipulate or a context was not meaningful. In the above example, the interviewer used prompts to suggest different representations to the student as she or he solved the problem. In all instances, these types of prompts aided the student in finding a workable representation/context. *Impediments to enacting the tasks* occurred five times and were evident only in Non-Anticipatory strategies. They were often overcome by substituting a different representation (e.g., pieces of paper instead of cubes; a rectangle instead of a circle) or context (e.g., a type of food more appealing to the child). A third-grade student displayed this factor before he used a Non-Anticipatory strategy to solve a problem involving five breadsticks and two sharers:

S: [reads problem . . . switches context of problem mid-read to match his own conception]. Breadsticks. Are those like . . . in lunch?
I: Yes. You've eaten those before?
S: Yeah . . . I don't know. I ate them . . . I didn't really like them.
I: You didn't?
S: No. When I ate them they tasted like . . . they tasted like something else.
I: Hmm. What did they taste like?
S: Like cardboard . . .
I: [laughs] Well, that isn't any good. What could we share that would taste good?
S: My mom cuts watermelon up all the time.
I: I like watermelon . . . I like watermelon a whole lot. What if we had five watermelons?
S: [counts out five papers for the watermelons; rips them all in half]. There. Ok five halves- you have five halves. And I have five halves.
I: Oh so we each get five halves? That makes sense to me.

Discussion

The results of the study support the notion that students with LD used similar strategies as described in existing frameworks documenting children's strategy used in equal sharing problems, although the range of strategy use was relatively compact. Put differently, students in the current study used mostly Non-Anticipatory strategies; they used Emergent Anticipatory strategies less frequently and did not use Anticipatory strategies found in previous research (Empson et al., 2005). In addition, the majority of strategies employed to solve problems was rudimentary in nature; students did not anticipate a coordination of sharers with the amount to be shared. These initial conceptions of fractions are consistent with the initial conceptions of fractions documented among students without LD in prior research and not atypical (Empson et al., 2005). Thus, the rudimentary nature of students' strategies is not necessarily emblematic of LD. Furthermore, the fact that students with LD's strategies for equal sharing reflect underdeveloped conceptions of fractions suggests that a focus on the development of conceptual understanding is critical to students' success. It is important to note that this was an introductory case study with 10 participants; results need to be confirmed with a larger sample. Caution should be used in extending the results found within the current study to all students with LD.

Implications for Practice

Prior research conducted with students with LD has proposed the nature of their conceptual gaps in understanding fractions originates from qualitatively different thinking. "Atypical" conceptions of fractions noted among adults (Lewis, 2010) suggested that students with LD conceive of fractions as something "taken away" or of the act of partitioning as an action as opposed to creating a fractional part of a whole. Van Garderen and Montague (2003) and van Garderen (2006) asserted that students with LD do not understand mathematical relationships between quantities while solving problems. In the current study, students with LD were presented equal sharing tasks where, in their activity, they used strategies indicative of a rudimentary conception of fractions as quantities. These strategies matched existing frameworks documenting student's strategies for solving equal sharing problems (Empson et al., 2005; Empson & Levi, 2011) and reflected a basic understanding of mathematical relationships between quantities. Importantly, we did not find any evidence to support the notion that students with LD conceived of fractions in a manner different from students not labeled with LD. Thus, our data suggest that the nature of the gap experienced by students with LD specific to their notions of fractions involves the relative sophistication of the strategies used to solve problems and not atypical conceptions (Siegler, 2007).

The level of sophistication of the strategies students employ to solve equal sharing tasks is connected to understanding the relationship between the act of partitioning and the resulting fractional quantities (Empson et al., 2005; Olive & Steffe, 2002; Pothier & Sawada, 1983; Steffe & Olive, 2009; Tzur, 1999). The development of this understanding is arguably directly related to opportunities students have to work with meaningful tasks that elicit and extend such conceptions within activity tied closely to current levels of understanding (Empson, 2003; Simon, Tzur, Heinz, & Kinzel, 2004). Results of the current study reveal that students with LD primarily used Non-Anticipatory strategies to create and manipulate fractional quantities, which suggests that multiplicative conceptions that provide the basis for a well-developed understanding of fractions are underdeveloped (Tzur, Xin, Si, Kenney, & Guebert, 2010). Thus, we argue that the nature of the performance gap in students' conceptual understanding of fractions is related to malleable factors.

Furthermore, the compacted range of strategies documented in the current study is not necessarily limited to LD students or indicative of intrinsic mathematics difficulties. Prior research suggests in fact that few students develop multiplicative conceptions of fraction in the absence of instruction that is focused on building this conceptual understanding. In Empson and colleagues' (2005) cross-sectional study, for example, the vast majority of students who had not participated in such instruction used Non-Anticipatory and Emergent Anticipatory strategies for equal sharing problems. Steffe (2007) estimated that at least 30% of students who have completed fifth grade do not have a multiplicative conception of fractions, a figure that includes a majority of students without LD.

Emerging research suggests that students' conceptual understanding of fractions can be cultivated through

> a) an adaptive form of instruction that builds on students' current funds of mathematical knowledge, b) tasks that make sense to students given *their* prior conceptions, and c) . . . representations that, *for the students*, meaningfully signify quantities linked to numbers and operations used in a task. (Tzur et al., 2010, p. 1; see also Empson, 2003; Hunt & Vasquez, 2013; Tzur, 1999)

Thus, the conceptions of fractions students with LD evidenced in the current study may be enriched by instruction that provides a focus on students "solv[ing] problems that are within [their] reach [while] grappling with key mathematical ideas that are comprehendible but not yet well formed" (Hiebert & Grouws, 2007, p. 387).

Teaching that is adaptive to students' emerging conceptual understanding may also reduce or eliminate some of the interfering factors found in the current study that seemed to influence students' strategies for equal sharing tasks (e.g., Hunt, Tzur, & Westenskow, under review). Students displayed difficulty moving forward in their conception of the mathematics of a problem because they employed keyword strategies, used representations that were difficult to manipulate, or looked to the teacher for validation of fraction conceptions during the process of solving problems. These factors are malleable and may be mitigated if not avoided altogether through instruction that focuses on supporting and extending children's conceptual understanding by engaging children in problem solving. For example, Moscardini (2010) found that children with LD in mathematics were able to solve word problems involving whole-number operations without prior explicit instruction in strategies and to develop their conceptual understanding of whole-number operations by this process.

Limitations and Future Research

Regardless of the encouraging results, there are several important limitations associated with this study that need to be acknowledged. First, this exploratory case study included 10 participants; results need to be confirmed with a larger sample. In addition, the current study did not take into account factors such as the role of prior or current instruction on students' employed strategies. It is important to note that the coding scheme did not take into account contextual factors such as the role of classroom, collaboration, or related factors, as the interviewers were done one-on-one out of the classroom environment. In other words, this is not the only way conceptual understanding can be documented or studied. This, the choice of coding scheme should not be interpreted as an all-encompassing definition of what conceptual understanding is in regard to fractions but rather as a focus on employed strategies and representations. A third limitation rests in the nature of the data collected. We examined the nature of students' conceptions of fractions as evidenced through employed problem-solving strategies in equal sharing problems. Thus, we have no information on students' interpretations of fractions in the context of other kinds of tasks. Also, we did not further explore any possible interaction of cognitive factors on problem-solving ability in the current study. It may be that these factors could affect employed strategies of students with LD initially and over time (Geary, Hoard, & Nugent, 2012). Finally, the level of prompting, while minimal (e.g., asking the student how their solution related to the problem, having them reflect on the need to share everything in the problem, asking them what their picture represented), varied by child and/or problem type, as called for by the clinical-interview method (Ginsburg, 1997). Yet, it is possible that strategies students used may have systematically appeared more advanced in some interviews than in others where the prompting was different. Future studies including the use of randomly selected comparison samples of students with and without LD would allow for the use of inferential statistics to examine the generalizability of our initial conclusions.

Replicating the findings with a larger sample of students in one school where real-time strategies of students with and without LD may work to expand current findings. Studies may also look at strategy use over time paired with instruction and provide a qualitative analysis of student–teacher interactions initially, as did the current study, as well as during instruction to provide insight into the nature of not only students' current conceptions of fractions but also their growth and development and how pedagogical mechanisms may affect strategies used and subsequent learning. Such research may also document any interactions or correlations between prompts and strategies utilized by students, problem type and strategies use, and/or cognitive profiles of students with developmental pathways and understandings.

Declaration of Conflicting Interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

References

Brantlinger, E., Jimenez, R., Klinger, J., Pugach, M., & Richardson, M. (2005). Qualitative studies in special education. *Exceptional Children*, *71*, 195–207.

Cawley, J. F., & Miller, J. H. (1989). Cross-sectional comparisons of the mathematical performance of students with LD: Are we on the right track toward comprehensive programming? *Journal of Learning Disabilities*, *23*, 250–254, 259.

Charles, K., & Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics*, *43*, 191–221.

Compton, D. L., Fuchs, L. S., Fuchs, D., Lambert, W., & Hamlett, C. (2012). The cognitive and academic profiles of reading and mathematics learning disabilities. *Journal of Learning Disabilities*, *45*, 79–95.

Davis, N., Cannistraci, C. J., Rogers, B. P., Gatenby, J. C., Fuchs, L. S., Anderson, A. W., & Gore, J. C. (2009). Aberrant functional activation in school age children at-risk for mathematical disability: A functional imaging study of simple arithmetic skill. *Neuropsychologia*, *47*, 2470–2479.

Empson, S. B. (2003). Low-performing students and teaching fractions for understanding: An interactional analysis. *Journal for Research in Mathematics Education*, *34*, 305–343.

Empson, S. B., Junk, D., Dominguez, H., & Turner, E. (2005). Fractions as the coordination of multiplicatively related quantities: A cross sectional study of student's thinking. *Educational Studies in Mathematics*, *63*, 1–18.

Empson, S. B., & Levi, L. (2011). *Extending children's mathematics: Fractions and decimals. Innovations in cognitively guided instruction*. Portsmouth, NH: Heinemann.

Fazio, L. K., & Siegler, R. S. (2013). Microgenetic learning analysis: A distinction without a difference. *Human Development*, *56*, 52–58.

Garofalo, J., & Lester, F. (1985). Metacognition, cognitive monitoring, and mathematical performance. *Journal for Research in Mathematics Education*, *16*, 163–176.

Geary, D. C., Hoard, M. K., & Nugent, L. (2012). Independent contributions of the central executive, intelligence, and in-class attentive behavior to developmental change in the strategies used to solve addition problems. *Journal of Experimental Child Psychology*, *113*, 49–65.

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge, UK: Cambridge University Press.

Hecht, S. A., & Vagi, K. J. (2010). Sources of group and individual differences in emerging fraction skills. *Journal of Educational Psychology*, *102*, 843–858.

Hecht, S. A., Vagi, K. J., & Torgesen, J. K. (2007). Fraction skills and proportional reasoning. In D. B. Berch & M. M. M. Mazzacco (Eds.), *Why is math so hard for some students?* (pp. 121–132). Baltimore, MD: Paul H. Brookes.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371–404). Greenwich, CT: Information Age.

Hunt, J. H., & Vasquez, E. (2013). An exploratory study of the effects of conceptual strategy intervention on knowledge of ratio equivalency for middle school students with mathematics learning disability. *The Journal of Special Education*, *47*, 1–11.

Leech, N. L., & Onwuegbuzie, A. J. (2007). An array of qualitative data analysis tools: A call for data analysis triangulation. *School Psychology Quarterly*, *22*, 557–584.

Lewis, K. E. (2010, June 29–July 2). Reconceptualizing mathematical learning disabilities: A diagnostic case study. In *Proceedings of the International Conference of the Learning Sciences (ICLS)*. (pp. 742–749)

Mazzocco, M. M. M., & Devlin, K. T. (2008). Parts and holes: Gaps in rational number sense in children with vs. without mathematical learning disability. *Developmental Science*, *11*, 681–691.

Mazzocco, M. M. M., & Kover, S. T. (2007). A longitudinal assessment of the development of executive functions and their association with math performance. *Child Neuropsychology*, *13*, 18–45.

Moscardini, L. (2010). "I like it instead of maths": How pupils with moderate learning difficulties in Scottish primary special schools intuitively solved mathematical word problems. *British Journal of Special Education*, *37*, 130–138.

National Center for Educational Statistics. (2009). *NAEP questions*. Retrieved from http://nces.ed.gov/nationsreportcard/itmrlsx/

National Mathematics Advisory Panel. (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. Washington, DC: National Academies Press.

Olive, J., & Steffe, L. P. (2002). The construction of an iterative fractional scheme: The case of Joe. *Journal of Mathematical Behavior*, *20*, 413–437.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, *14*, 307–317.

Siegler, R. S. (2005). Children's learning. *American Psychologist*, *60*, 769–778.

Siegler, R. S. (2007). Cognitive variability. *Developmental Science*, *10*, 104–109.

Simon, M., Saldanha, L., McClintock, E., Karagoz Akar, G., Watanabe, T., & Ozgur Zembat, I. (2010). A developing approach to studying students' learning through their mathematical activity. *Cognition and Instruction*, *28*, 70–112.

Simon, M., Tzur, R., Heinz, K., & Kinzel, M. (2004). Explicating a mechanism for conceptual learning: Elaborating the construct of reflective abstraction. *Journal for Research in Mathematics Education*, *35*, 305–329.

Steffe, L. P. (2002). A new hypothesis concerning children's fraction knowledge. *Journal of Mathematical Behavior*, *102*, 1–41.

Steffe, L. P. (2007, April). *Problems in mathematics education*. Paper presented for the Senior Scholar Award of the Special Interest Group for Research in Mathematics Education (SIG-RME) at the annual conference of the American Educational Research Association, Chicago, IL.

Steffe, L. P., & Olive, J. (2009). *Children's fractional knowledge*. New York, NY: Springer.

Streefland, L. (1993). Fractions: A realistic approach. In T. P. Carpenter, E. Fennema, & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 289–326). Hillsdale, NJ: Lawrence Erlbaum.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting that learning. *Journal for Research in Mathematics Education*, *30*, 390–416.

Tzur, R., Johnson, H. L., McClintock, E., & Risley, R. (2012, November 1). *Culturally-mathematically relevant pedagogy (CMRP): Fostering urban English language learners' multiplicative reasoning*. Paper presented at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Kalamazoo, MI.

Tzur, R., Xin, Y. P., Si, L., Kenney, R., & Guebert, A. (2010, April/May). *Students with learning disability in math are left behind in multiplicative reasoning? Number as abstract composite unit is a likely "culprit."* Paper presented at the American Educational Research Association, Denver, CO.

van Garderen, D. (2006). Spatial visualization, visual imagery, and mathematical problem solving of students with varying abilities. *Journal of Learning Disabilities*, *39*, 496–506.

van Garderen, D., & Montague, M. (2003). Visual-spatial representation, mathematical problem solving, and students of varying abilities. *Learning Disabilities Research & Practice*, *18*, 246–254.

Vukovic, R. K. (2012). Mathematics difficulty with and without reading difficulty: Findings and implications from a four-year longitudinal study. *Exceptional Children*, *78*, 280–300.