WHY IS ALGEBRA HARD?

TWO (OF MANY) RELATED OBSTACLES TO UNDERSTANDING

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CADRE Panel, 2010

Slides available at
`www.edc.org/cmeproject`

Outline

From a popular text ($\sim$ 1980)

"Factoring Pattern for $x^2 + bx + c$, c Negative"

Factor. Check by multiplying factors. If the polynomial is not factorable, write "prime."

1. $a^2 + 4a - 5$	**2.** $x^2 - 2x - 3$	**3.** $y^2 - 5y - 6$
4. $b^2 + 2b - 15$	**5.** $c^2 - 11c - 10$	**6.** $r^2 - 16r - 28$
7. $x^2 - 6x - 18$	**8.** $y^2 - 10c - 24$	**9.** $a^2 + 2a - 35$
10. $k^2 - 2k - 20$	**11.** $z^2 + 5z - 36$	**12.** $r^2 - 3r - 40$
13. $p^2 - 4p - 21$	**14.** $a^2 + 3a - 54$	**15.** $y^2 - 5y - 30$
16. $z^2 - z - 72$	**17.** $a^2 - ab - 30b^2$	**18.** $k^2 - 11kd - 60d^2$
19. $p^2 - 5pq - 50q^2$	**20.** $a^2 - 4ab - 77b^2$	**21.** $y^2 - 2yz - 3z^2$
22. $s^2 + 14st - 72t^2$	**23.** $x^2 - 9xy - 22y^2$	**24.** $p^2 - pq - 72q^2$

From a published text (2010)

To factor a trinomial of the form $ax^2 + bx + c$ where $a > 0$, follow these steps:

$$\begin{array}{c|c|c} A & B & C \\ \hline F & H & D \\ \hline G & I & E \end{array}$$

1. Identify the values of a, b, c. Put a in Box A and c in Box B. Put the product of a and c in Box C.
2. List the factors of the number from Box C and identify the pair whose sum is b. Put the two factors you find in Box D and E.
3. Find the greatest common factor of Boxes A [sic] and E and put it in box G.
4. In Box F, place the number you multiply by Box G to get Box A.

From a published text (2010)

$$\begin{array}{c|c|c} A & B & C \\ \hline F & H & D \\ \hline G & I & E \end{array}$$

5 In Box H, place the number you multiply by Box F to get Box D.

6 In Box I, place the number you multiply by Box G to get Box E.

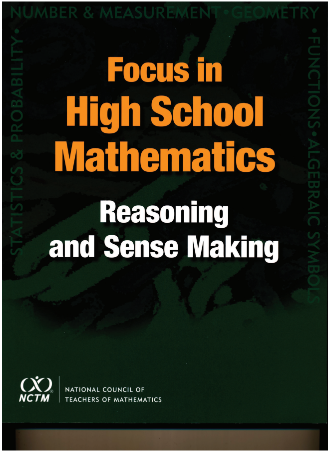
Solution: The binomial factors whose product gives the trinomial are $(Fx + I)(Gx + H)$.

Why is Algebra Hard?

- **One Reason**: High school algebra lacks the mathematical coherence that exists in algebra as a scientific discipline.
- Lack of coherence makes things hard.

- **Another Reason**: That coherence comes from certain algebraic habits of mind that receive little attention in most programs.
- Developing habits takes time and purpose.

NCTM: Focus in High School Mathematics



"A number of documents have been produced over the past few years providing detailed analyses of the topics that should be addressed in each course of high school mathematics. . . ."

"[FHSM] takes a somewhat different approach, proposing curricular emphases and instructional approaches that make reasoning and sense making foundational to the content that is taught and learned."

(Hooray)

Focus in High School Mathematics: Algebra

Key elements of reasoning and sense making with algebraic symbols include

- **Meaningful use of symbols**.
- **Mindful manipulation**.
- **Reasoned solving**.
- **Connecting algebra with geometry**.
- **Linking expressions and functions**.

Common Core: Mathematical Practices

Eight attributes of mathematical proficiency:

- **Make sense of complex problems and persevere in solving them.**
- **Reason abstractly and quantitatively.**
- **Construct viable arguments and critique the reasoning of others.**
- **Model with mathematics.**
- **Use appropriate tools strategically.**
- **Attend to precision.**
- **Look for and make use of structure.**
- **Look for and express regularity in repeated reasoning.**

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Mathematical Habits of Mind

General Mathematical Habits

- Performing thought experiments
- Finding and explaining patterns
- Using precise language
- Creating and using representations
- Generalizing from examples
- Expecting mathematics to make sense

Mathematical Habits of Mind

Analytic/Geometric Habits of Mind

- Reasoning by continuity
- Seeking geometric invariants
- Looking at extreme cases
- Passing to the limit
- Modeling geometric phenomena with continuous functions

Mathematical Habits of Mind

Algebraic Habits of Mind

- Seeking and expressing regularity in repeated calculations
- “Delayed evaluation”—seeking form in calculations
- “Chunking”—changing variables in order to hide complexity
- Reasoning about and picturing calculations and operations
- Extending operations to preserve rules for calculating
- Purposefully transforming and interpreting expressions
- Seeking and specifying structural similarities

Example 1: Factoring in Algebra 1

Factoring monic quadratics:

"Sum-Product" problems

$$x^2 + 14x + 48$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

so...

Find two numbers whose sum is 14 and whose product is 48.

$$(x + 6)(x + 8)$$

Example 1: Factoring in Algebra 1

What about this one?

$$49x^2 + 35x + 6$$

$$\begin{aligned} 49x^2 + 35x + 6 &= (7x)^2 + 5(7x) + 6 \\ &= \clubsuit^2 + 5\clubsuit + 6 \\ &= (\clubsuit + 3)(\clubsuit + 2) \\ &= (7x + 3)(7x + 2) \end{aligned}$$

Example 1: Factoring in Algebra 1

What about this one?

$$6x^2 + 31x + 35$$

$$\begin{aligned}
6(6x^2 + 31x + 35) &= (6x)^2 + 31(6x) + 210 \\
&= \clubsuit^2 + 31\clubsuit + 210 \\
&= (\clubsuit + 21)(\clubsuit + 10) \\
&= (6x + 21)(6x + 10) \\
&= 3(2x + 7) \cdot 2(3x + 5) \\
&= 6(2x + 7)(3x + 5) \quad \textit{so} \ldots
\end{aligned}$$

$$6(6x^2 + 31x + 35) = 6(2x + 7)(3x + 5)$$

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\end{aligned}$$

$$6x^2 + 31x + 35 = (2x + 7)(3x + 5)$$

Example 1: Factoring in Algebra 1

- This technique is perfectly general and can be used to transform a polynomial of any degree into one whose leading coefficient is 1.
- And it fits into the larger landscape of the *theory of equations* that shows how to use similar transformations to
  - remove terms
  - transform roots
  - derive “formulas” for equations of degree 3 and 4
  - extend the notion of *discriminant* to higher degrees
  - preview ideas from Galois theory

Other Examples Where This Habit Is Useful

- Completing the square and removing terms
- Solving trig equations
- Analyzing conics and other curves
- All over calculus
- Interpreting results from a CAS

$\vdots$

Some Thorny Topics in Elementary Algebra

1. Students have trouble expressing generality with algebraic notation.
2. This is especially prevalent when they have to set up equations to solve word problems.
3. Many students have difficulty with slope, graphing lines, and finding equations of lines.
4. Building and using algebraic functions is another place where students struggle.

This list looks like a collection of disparate topics—using notation, solving word problems, ...

Some Thorny Topics in Elementary Algebra

But if one looks underneath the topics to the mathematical habits that would help students master them, one finds a remarkable similarity:

A key ingredient in such a mastery is the reasoning habit of seeking and expressing regularity in repeated calculations.

- This habit manifests itself when one is performing the same calculation over and over and begins to notice the "rhythm" in the operations.
- Articulating this regularity leads to a generic algorithm, typically expressed with algebraic symbolism, that can be applied to any instance and that can be transformed to reveal additional meaning, often leading to a solution of the problem at hand.

EXAMPLE 1: THE DREADED ALGEBRA WORD PROBLEM

Think about how hard it is for students to set up an equation that can be used to solve an algebra word problem. Some reasons for the difficulties include reading levels and unfamiliar contexts. But there has to be more to it than these surface features.

Consider, for example, the following two problems.

Example 1: The Dreaded Algebra Word Problem

1. The driving distance from Boston to Chicago is is 990 miles. Rico drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. For how many hours is Rico on the road?

2. Rico drives from Boston to Chicago at an average speed of 50mph and returns at an average speed of 60mph. Rico is on the road for 36 hours. What is the driving distance from Boston to Chicago?

The problems have identical reading levels, and the context is the same in each. But teachers report that many students who can solve problem 1 are baffled by problem 2.

Example 1: The Dreaded Word Problem

This is where the reasoning habit of “expressing the rhythm” in a calculation can be of great use. The basic idea:

- Guess at an answer to problem 2, and
- check your guess as if you were working on problem 1, *keeping track of your steps*.

The purpose of the guess is not to stumble on (or to approximate) the correct answer; rather, it is to help you construct a “checking algorithm” that will work for any guess.

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Example 1: The Dreaded Word Problem

Problem 2: He drives over at an average speed of 50mph and returns at an average of 60mph. He's on the road for 36 hours. What is the driving distance?

So, you take several guesses until you are able to express your checking algorithm in algebraic symbols. For example:

- Suppose the distance is 1000 miles.
- How do I check the guess of 1000 miles? I divide 1000 by 50. Then I divide 1000 by 60. Then I add my answers together, to see if I get 36. I don't.
- So, check another number—say 950. 950 divided by 50 plus 950 divided by 60. Is that 36?
- No, but a general method is evolving that will allow me to check *any* guess.

Example 1: The Dreaded Word Problem

- My guess-checker is

$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} \stackrel{?}{=} 36$$

- So my *equation* is

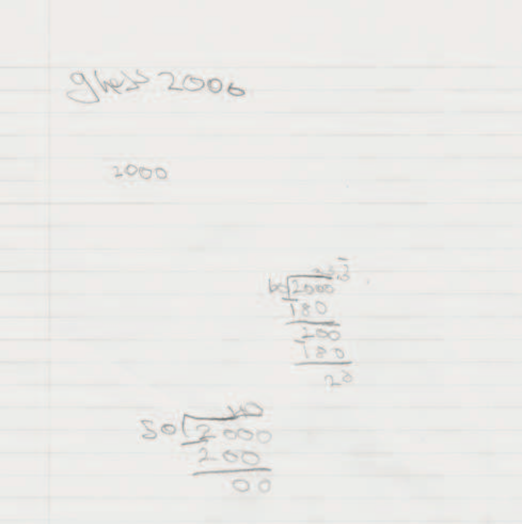
$$\frac{\text{guess}}{50} + \frac{\text{guess}}{60} = 36$$

or, letting x stand for the unknown correct guess,

$$\frac{x}{50} + \frac{x}{60} = 36$$

Example 1: The Dreaded Word Problem

Here's some student work that shows how the process develops:



Example 1: The Dreaded Word Problem

$40 + 33.\overline{3} = 73.\overline{3}$ hours

guess: 1500 mph

$$\begin{array}{r} 25 \\ 60\overline{)1500} \\ \underline{120} \\ 300 \\ \underline{300} \\ 0 \end{array}$$

25

1500

$$\begin{array}{r} 30 \\ 50\overline{)1500} \\ \underline{150} \\ 00 \end{array}$$

$25 + 30 = 55$ hrs

(guess) ÷ 60) + (guess ÷ 50) = 36

$(x \div 60) + (x \div 50) = 36$

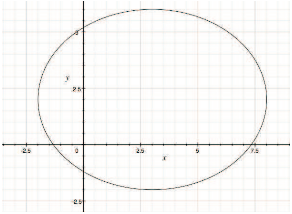
EXAMPLE 2: EQUATIONS FOR LINES

The phenomenon was first noticed in precalculus ...
Graph

$$16x^2 - 96x + 25y^2 - 100y - 156 = 0$$

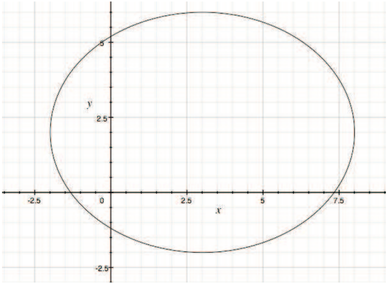
$$16x^2 - 96x + 25y^2 - 100y - 156 = 0 \Rightarrow \frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1 \Rightarrow$$



Example 2: Equations for lines

$$\frac{(x-3)^2}{25} + \frac{(y-2)^2}{16} = 1$$



Is $(7.5, 3.75)$ on the graph?

This led to the idea that “equations are point testers.”

Example 2: Equations for lines

- Suppose a student, new to algebra and with no formulas in tow, is asked to find the equation of the vertical line ℓ that passes through $(5, 4)$.
- Students can draw the line, and, just as in the word problem example, they can guess at some points and check to see if they are on ℓ.
- Trying some points like $(5, 1)$, $(3, 4)$, $(2, 2)$, and $(5, 17)$ leads to a generic guess-checker:

  To see if a point is on ℓ, you check that its x-coordinate is 5.

- This leads to a guess-checker: $x \stackrel{?}{=} 5$ and the equation

$$x = 5$$

EXAMPLE 2: EQUATIONS FOR LINES

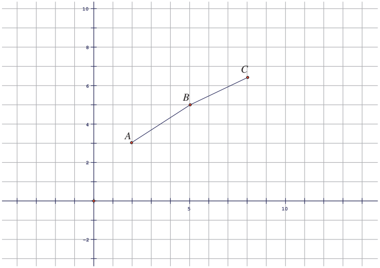
- What about lines for which there is no simple guess-checker? The idea is to find a geometric characterization of such a line and then to develop a guess-checker based on that characterization. One such characterization uses *slope*.

- In first-year algebra, students study slope, and one fact about slope that often comes up is that three points on the coordinate plane, not all on the same vertical line, are collinear if and only if the slope between any two of them is the same.

Example 2: Equations for lines

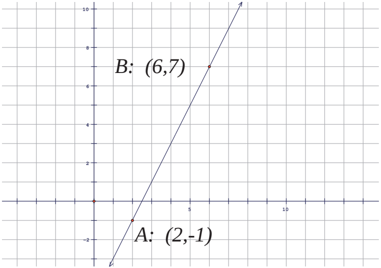
If we let $m(A, B)$ denote the slope between A and B (calculated as change in y-height divided by change in x-run), then the collinearity condition can be stated like this:

Basic assumption: A, B, and C are collinear $\Leftrightarrow m(A, B) = m(B, C)$



Example 2: Equations for lines

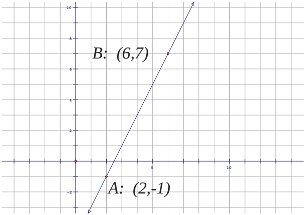
What is an equation for $\ell = \overleftrightarrow{AB}$ if $A = (2, -1)$ and $B = (6, 7)$?



Try some points, keeping track of the steps...

Example 2: Equations for lines

- $A = (2, -1)$ and $B = (6, 7)$
- $m(A, B) = 2$



- Test $C = (3, 4)$: $m(C, B) = \frac{4-7}{3-6} \stackrel{?}{=} 2 \Rightarrow$ Nope
- Test $C = (5, 5)$: $m(C, B) = \frac{5-7}{5-6} \stackrel{?}{=} 2 \Rightarrow$ Yup
- The "guess-checker?" Test $C = (x, y)$: $m(C, B) = \frac{y-7}{x-6} \stackrel{?}{=} 2$

And an equation is $\frac{y-7}{x-6} = 2$

Other Examples Where This Habit Is Useful

- Finding lines of best fit
- Fitting functions to tables of data
- Deriving the quadratic formula
- Establishing identities in Pascal's triangle
- Using recursive definitions in a CAS or spreadsheet

$\vdots$

CONCLUSIONS

- Organizing precollege algebra solely around lists of topics and low-level skills hides the essential coherence of the subject.
- Such organizations lead to implementations that don't convey the spirit of algebra—they make algebra hard.
- An organization around algebraic habits of mind has the potential to
  - bring coherence, parsimony, and simplicity to the subject,
  - create programs that are faithful to algebra,
  - eliminate unnecessary obstacles to understanding,
  - help students develop mathematical habits that are useful in and out of school.