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Competencies and behaviors observed when students solve geometry proof problems: an interview study with smartpen technology

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Abstract
Decades of research have established that solving geometry proof problems is a challenging endeavor for many students. Consequently, researchers have called for investigations that explore which aspects of proving in geometry are difficult and why this is the case. Here, results from a set of 20 interviews with students who were taught proof in school geometry are reported. Students who earned A or B course-grades in the proof unit(s) were asked to share their thinking aloud while solving two proof tasks using smartpens. Student thinking was analyzed for two subgroups—students who were successful with both proofs ($n=7$) and who were unsuccessful with both proofs ($n=13$). Large differences were observed in how often students in the two groups exhibited certain competencies and behaviors. The largest gaps occurred in the ways in which students attended to the proof assumptions, attended to warrants in their proofs, and demonstrated logical reasoning.

Keywords Geometry · Proof · Problem solving · Student thinking · Smartpen technology · Competencies · Pronouns

1 Introduction

Although geometry has historically been a starting point to teach and learn mathematical proof in secondary mathematics (Reiss, Hellmich, & Reiss, 2002), the teaching of proof in geometry has been called a failure in almost all countries (Balacheff, 1988).[1] This claim is supported by an abundance of evidence (see, e.g., Healy & Hoyles, 1998; Reiss, Klieme, & Heinze, 2001; Senk, 1985). Yet, when we view proof construction as a problem-solving endeavor (Schoenfeld, 1992; Weber, 2001), these results are unsurprising. After all, problem solving is demanding on both teachers and students (Schoenfeld, 1992), and geometry proof problem solving, in particular, is hard (Koedinger & Anderson, 1990).

The widespread lack of success in this area was acknowledged in Battista's (2007) review of school geometry research, where he posed several unanswered questions related to students' learning of proof in geometry, including: Why do students have so much difficulty with proof? What components of proof are difficult for students and why? and How can proof skills best be developed in students? (pp. 887–888). Ten years later, speaking back to these questions in their research review, Sinclair, Cirillo, and deVilliers (2017) concluded that while some researchers have attempted to address these questions, more research is needed on students' development of geometry proof skills and their understanding of the nature of proof. To make progress on these unanswered questions, we sought to identify competencies displayed by high-attaining students as they attempted to solve geometry proof problems using smartpens.

2 Theoretical framework

2.1 Competencies for problem solving

Much of Schoenfeld's work on problem solving has been situated in the content domain of geometry where he often focused on "geometric proof problems" to illustrate his ideas (see, e.g., Schoenfeld, 1985, 1992). In a discussion of resources that can contribute to an individual's problem-solving performance in a particular mathematical

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[1] Our interpretation of this claim is not that there are some countries teaching proof well, but, rather, that not enough documentary evidence to support a claim that teaching proof is a failure in *all* countries exists.

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domain for problem solving, Schoenfeld (1985) framed the notion of "relevant competencies" by asking: "What mathematical knowledge does the individual have that he or she might be able to bring to bear on a given problem?" (p. 59). Following Döhrmann, Kaiser, and Blömeke (2012), who connected mathematics competencies to the content and cognitive domains described in TIMSS, here we draw on the TIMSS framework to operationalize our use of the term "competencies."

Our operationalization of "competencies" for geometry proof problem solving encompasses the three cognitive domains outlined in the TIMSS 2015 Assessment Framework. In their description of the framework, Mullis and Martin (2013) suggested that a range of cognitive skills could be described within the domains of *knowing*, *applying*, and *reasoning*. Due to space limitations, in summarizing these three domains, we focus on Mullis and Martin's examples and descriptors that are situated in the content domain of geometry.

Knowing includes recalling definitions and geometric properties, classifying shapes by common properties, retrieving information from various sources, and recognizing shapes and entities that are mathematically equivalent (e.g., different orientations of simple geometric figures). A knowledge base is important because facility in reasoning about mathematical situations depends on familiarity with mathematical concepts, symbolic representation, and spatial relationships.

Applying involves the application of mathematics in a range of contexts. In this cognitive domain, students apply mathematical knowledge of facts, skills, and procedures. *Applying* includes determining appropriate strategies and methods of solution and implementing strategies to solve problems involving familiar mathematical concepts and procedures.

Reasoning involves logical, systematic thinking, including making logical deductions based on specific assumptions and rules. *Reasoning* includes the following cognitive skills: analyzing relationships, integrating and synthesizing different elements of knowledge, evaluating problem-solving strategies and solutions, drawing conclusions based on information and evidence, and justifying a strategy or solution.

Our use of the term *competencies*, therefore, includes cognitive skills from the mathematics cognitive domains of *knowing, applying,* or *reasoning*. Competencies may be, and often are, comprised of some combination of skills from these domains (e.g., *knowing* the definition of congruent triangles and *applying* this knowledge to write a congruence statement about a particular pair of triangles).

2.1.1 Research on the role of knowledge in problem solving

Some researchers have explored the role of knowledge in solving proof problems. For example, Kantowski (1977) found that students who committed geometry definitions and theorems to memory more easily activated these justifications for their proofs. Definitions and theorems not committed to memory were used less often even though students were given a reference sheet of these ideas. In contrast to Kantowski, who discussed domain knowledge in a binary way (i.e., students either had the knowledge or they did not), Schoenfeld (1985) described knowledge as sometimes being "shaky" (p. 57). In other words, some of the "knowledge" that an individual brings to a problem situation may be incorrectly remembered or simply wrong, and therefore has the potential to sabotage a solution. Still, Mamona-Downs and Downs (2005) claimed that students can do well with a knowledge test but then perform poorly with the associated problem-solving tasks. In other words, simply having the knowledge is insufficient. They suggested that knowing how to apply one's knowledge is at least as important as actually possessing the knowledge itself.

2.2 Competencies for geometry proof problem solving

Problem solving expertise is often determined by studying "experts"—individuals or groups of individuals who exhibit superior performance in problem solving—such as mathematicians or students who excel in mathematics (Weber & Leikin, 2016). For example, Koedinger and Anderson (1990) claimed that "detailed study of successful performance in difficult task domains can provide a strong basis for understanding the processes of problem solving and the nature of thought in general" (p. 511). Through their observations of "experts" engaged in geometry proof problem solving, Koedinger and Anderson (1990) found that prior to writing up the details of their proofs, experts tended to quickly and accurately develop an abstract proof-plan that skips many of the steps required in a full proof. In other words, experts first applied global thinking (i.e., considered the "big picture") rather than local thinking (i.e., worked on one step at a time) at the start of the process. This is consistent with findings from Cai's (1994) study of problem solving in geometry: more-experienced participants spent the majority of their time on orientation and organization, while less-experienced participants spent the majority of their time on execution (i.e., doing rather than thinking or planning).

In addition to considering the general approach of experts, specific competencies for proving in geometry, such as the ability to read and work with diagrams (Sinclair, Pimm, & Skelin, 2012) and draw valid conclusions

(Cirillo & Hummer, 2019), have been identified in the literature. Problematically, students often apply a perceptual proof scheme (Harel & Sowder, 2007) and incorrectly draw conclusions based on what looks to be true in the diagram rather than applying reasoning to the "Given" information (Cirillo & Hummer, 2019). The idea that we should and can explicitly teach important competencies, for example, how to read and work with diagrams and draw valid conclusions from the proof assumptions, is related to Senk's (1985) suggestion that we should aim to teach students to begin a chain of reasoning. Success with proof also requires a basic understanding of the role of logic in proving.

Definitions of proof tend to, at least implicitly, suggest a logical structure to the proof argument. For example, Movshovitz-Hadar's (2001) description of proof, which is the one adopted for this study, implicitly communicates that a proof is a logical argument:

> …an ideal mathematical proof displays in a systematic way a finite sequential set of statements that leads from definitions, axioms (i.e., statements the truth of which is unquestioned in a given theory) and theorems (i.e., statements the truth of which has already been proved) to a conclusion, in such a way that as long as the axioms are accepted and the definitions are agreed upon, the conclusion is inevitable and its validity must be recognized. (p. 585).

Logic can be observed in the chains of reasoning, or sub-arguments, that make use of two or more definitions or theorems (e.g., If lines are perpendicular, then angles formed are right angles; if angles are right angles, then they are congruent). Sub-arguments—"branches" or portions of the larger proof argument—are structured by the Law of Syllogism (Cirillo et al., 2017). As students plan or work through the details of a proof, their use of logic may be evident in a variety of ways. For example, students' deductive explanations might contain logical connectives such as "because" and "so" (Donaldson, 1986). Movshovitz-Hadar's description of proof also highlights the importance of knowledge about warrants used in a proof (i.e., definitions, axioms, and theorems).

3 Research goal and research questions

We sought to gain insight into the competencies enacted while students engaged in geometry proof problem solving. In this study, "competencies" are comprised of skills from the three cognitive domains of *knowing*, *applying*, and *reasoning* described in Sect. 2.1, including combinations of these skills. To identify such competencies, high-attaining students were asked to share their thinking aloud while using smartpens to solve two geometry proof problems. The data also prompted us to consider behaviors, which we view as being different from competencies. We posed the primary research question: What competencies were observed during clinical interviews of high-attaining students as they worked to solve geometry proof problems using smartpens? A secondary question was also explored: What behaviors were observed, particularly in the absence of proof-related problem-solving competencies?

4 Methods

4.1 Research context

The study reported here is part of the research project: *Proof in Secondary Classrooms: Decomposing a Central Mathematical Practice* (*PISC;* PI: Cirillo). The goal of the PISC project is to better understand the difficulties involved in the teaching and learning of proof in secondary geometry and to develop a new and improved intervention to address these challenges.

Students who earned high marks (grades of A or B) in the geometry proof unit(s) were selected for individual clinical interviews for this sub-study. The rationale for interviewing students with high marks was to understand what high-attaining students were taking away from the proof unit(s). We hypothesized that challenges identified for high-performing students would likely be challenges for all students. Because past studies have shown that even high-attaining students struggle with non-routine as well as routine proof problems in geometry (see, e.g., Healy & Hoyles, 1998; Cirillo, 2018), two proof tasks that, in theory, should have been familiar to the students, were selected. We chose triangle congruence proof as a topic for exploration because it is considered to be a central concept in school geometry.

4.2 Setting and participants

Four sub-urban racially and economically diverse school districts in the mid-Atlantic region of the United States participated in the PISC interview study. Interviews were conducted during Years 2 and 4 of the study. Criteria for participant selection included: (1) students earned an A or a B in the proof unit(s); (2) students were identified by their teacher as people who would be willing to share their thinking aloud during the interview; (3) students completed the full interview protocol in the allotted time; and (4) there were no technology glitches during the data collection.[2] This process reduced the sample size from 31 students interviewed to 23.

[2] A small number of data glitches occurred with the smartpen technology.

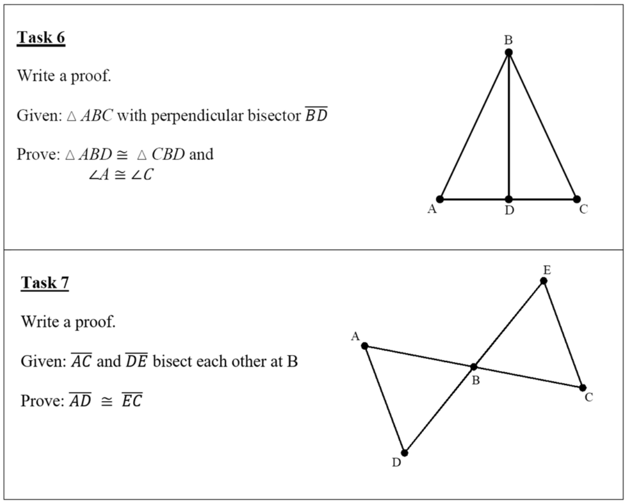


Fig. 1 The two proof tasks used in the study

Participants were enrolled in a course that addressed proof in geometry. Because coverage of this topic varies by district and curriculum program, the age and grade levels of participants varied. More specifically, younger students in the study were enrolled in Grade 8 Honors Geometry; while many of the older students were enrolled in an integrated math course that covered proof in year 2 or 3 of high school. Thus, participating students spanned Grades 8–11 (ages 13–17). The extent to which proof was covered varied by site, but, minimally, all participants had instruction in triangle congruence proofs.

4.3 Interview protocol and data collection

The full interview protocol consisted of seven items. The first item was a simple "warm-up" task about geometric notation. The next four tasks, which targeted different aspects of proof, were adapted from Cirillo and Herbst (2011). The last two tasks were selected for this analysis because they were the only full-proof tasks (see Fig. 1). Students spent an average of 7.28 min on Task 6 and 5.95 min on Task 7. They were asked to read each task aloud to ease them into thinking aloud and to guarantee they had read the "Given" statements.

Smartpen technology (i.e., *Livescribe* pens) was used to audio-record students' explanations of their thinking and capture their pen strokes as they worked through the proofs. This methodology allowed us to capture student thinking in the form of verbal explanations and simultaneous diagram markings and other written work.

4.4 Data and analysis

Smartpen data were digitized to create a "pencast," or video, that simultaneously replays each student's handwriting and the audio-recording (*Livescribe*, 2012). As the video plays, "active ink" is displayed on the screen in green in a way that syncs with the user's pen strokes. The inactive ink can also be seen in the video image, but it is grayed out until it becomes "live."[3] All interviews were transcribed. Hardcopies of the students' work, recorded on special paper required for the smartpen, were catalogued.

Prior to analyzing the smartpen data, students' final proofs were quantitatively scored from the paper hardcopies in ways that followed Senk's (1983) methods. Specifically, we adapted Senk's full-proof rubrics,[4] scoring each proof on a scale of 0–4. Following Senk's approach, if students scored a 3 or a 4 on a proof, they were considered to be **S**uccessful with the **P**roof task (abbreviated as SP). Students who scored less than 3 were considered **N**ot successful with the **P**roof task (abbreviated as NP). Students scored a 3 if their proof steps followed logically from previous ones but contained minor errors (e.g., notational errors, vocabulary, or names of reasons).

Students were sorted into three categories: those who were not successful on either task ($n=13$); those who were successful in solving one of the two tasks ($n=3$); and those who were successful in solving both tasks ($n=7$). Because we expected that the three partially successful provers would demonstrate a combination of results that overlapped with the results of SPs and NPs, we did not include their data in the next round of analysis.[5] This decision resulted in a data set comprising of two Proof Task Interviews (PTIs) from 20 students. The units of analysis are the individual proof task interviews, resulting in 40 units of analysis.

Analyses of the interview data occurred in phases. We used constant comparative analysis (Boeije, 2002) to develop a codebook and followed Creswell and Poth's (2016) procedures for reliability of intercoder agreement in qualitative research. Specifically, the research team, consisting of the two authors and two other team members, watched the PTI pencasts for six participants for both Tasks 6 and 7. Beginning with SPs, codes were developed for observed competencies exhibited through spoken and written work as students engaged in geometry proof problem solving. We coded all 20 PTIs using the initial 23 codes. Due to the absence of many of the competencies identified in SP

[3] This is useful information for viewing figures in the Findings.

[4] Rubrics were included in Senk's (1983) dissertation.

[5] This hypothesis was confirmed when we coded these data later on. We chose not to include these details because they over-complicate the reporting of the findings and were not very enlightening.

data, we then used open coding to develop additional codes for analyzing NP data. Because a low number of previously identified or other competencies were observed in the NP data, we, instead, developed codes to describe the most common behaviors that were observed (e.g., student reads the “Given” statement incorrectly). An initial reliability check was conducted by having three researchers code three PTIs. Although a high level of agreement was attained, the process resulted in some refinements of the codebook. Next, two researchers independently coded six additional PTIs. After garnering an 86.59% interrater reliability and reconciling incongruent decisions, the second author coded the remaining data. This iterative process resulted in a total of 45 possible codes. The final phase of analysis involved looking for patterns and themes across the data.

5 Findings

In reporting the findings, we first describe the most prevalent competencies observed for the students who were **S**uccessful with both **P**roof tasks (SPs; $n = 7$). We also describe one notable behavior identified in the SP data. Coding was done in a binary way such that evidence was either found or not found for any given code for each of the 40 units of analysis. This allowed us to calculate percentages of occurrences for each code within each task for each student. Because multiple competencies were exhibited by a high percentage of SPs, we chose a threshold of 60% for the occurrences that would be reported. This allowed us to report out most of the features of SPs’ work, leaving out only a few codes which occurred less frequently. In order to efficiently present the most interesting findings, we combined some of the codes. Reporting on five competencies and one behavior occurring at least 60% of the time for SPs yielded six findings. The main coding categories are described at the beginning of each section and, the first five also appear in Table 2.

Regarding competencies and behaviors of students who were **N**ot successful with either **P**roof task (NPs; $n = 13$), the data were more inconsistent. We first compare the results of NPs with SPs by reporting frequencies (as a percentage of total occurrences for each PTI) with respect to five competencies explored in the SP group. Table 2 provides a summary of the competencies and their frequencies for both groups. We then share three additional findings related to NPs’ behaviors.

5.1 Observations of students who were successful with the proofs

5.1.1 Students productively attended to the “Given” information

Productively attending to the “Given” information means that students, in some way, referred to the “Given” statement (versus ignoring it) and used it to make valid deductions. All seven SPs made productive and explicit use of the “Given” information for both tasks (i.e., 100% of the time). They did so either as they planned or began working on a proof. They explicitly identified the relevant mathematical objects from the assumptions (i.e., the “Given”). Examples are provided below for Tasks 6 and 7 (see Fig. 1) from P18 and P14,[6] respectively.

> P18: So, the Given—you have triangle ABC and perpendicular bisector BD, that’s Given. So, now I know that BD is perpendicular to AC because of the definition of perpendicular bisector, and I know that also D is the midpoint.[7]
> P14: First thing we know is that ABC and DE bisect each other at B. Why? Because it’s the Given. Next, well you know that B, B is the midpoint. Why? Because definition of line segment bisector…

In Task 6, after reading the “Given” aloud, P18 restated the assumption explicitly, saying, “So the Given…that’s Given.” He then applied the definition of perpendicular bisector to draw conclusions about perpendicular line segments and the midpoint of $\overline{AC}$ In Task 7, P14 read the “Given” aloud, and then restated it in his own words substituting $\overline{AC}$ with $\overline{ABC}$. He then said, “Why? Because it’s given.” This is evidence that he used the “Given” to deduce his first claim in the next statement.

Each proof task included some type of bisector in the “Given.” SPs were clear about which bisector they were working with 100% of the time (e.g., line segment bisector). They explicitly applied this knowledge, indicating what specifically was being bisected, 93% of the time.

5.1.2 Students used the diagram as a resource

Using the diagram as a resource means that students used the diagram in observable ways, such as marking it after making an accurate deduction, using it as a planning tool, or making valid assumptions about the diagram (e.g., noticing

6 P14 refers to the 14th participant. P14–P20 were the SPs, and P1–P13 were the NPs.

7 We “cleaned” up the transcripts slightly by removing “ums,” “uhs,” and so forth, for ease of reading.

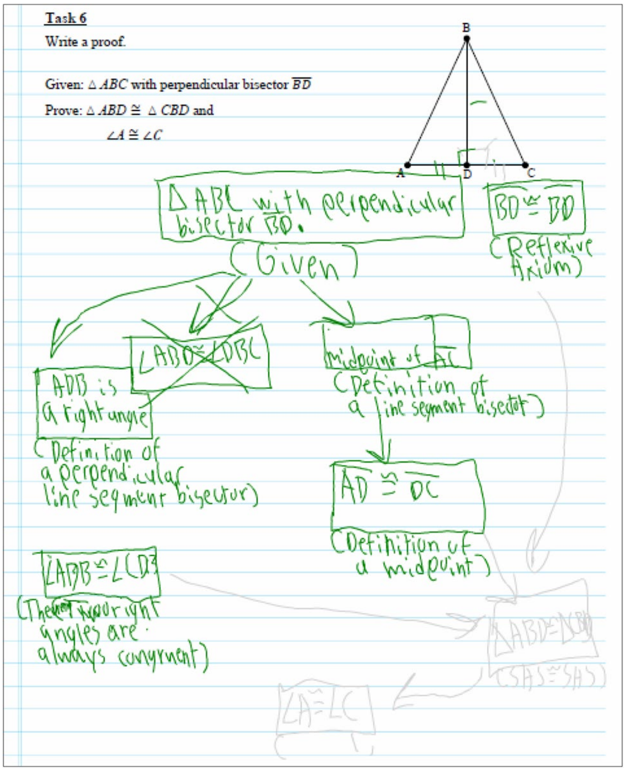


Fig. 2 P15's proof of Task 6

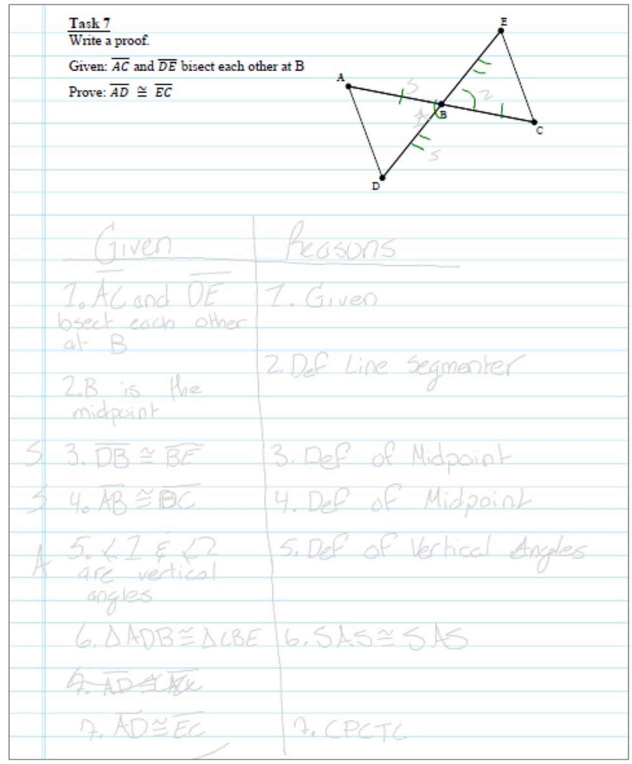


Fig. 3 P20's work on Task 7 prior to writing the proof details

vertical angles). All SPs accurately marked the diagrams for both proofs (i.e., 100% frequency). The smartpen technology enabled us to see noticeable differences in the ways this occurred. For Task 6, SPs marked the diagram in two distinct ways. Either they worked through the details of the proof, marking off congruent parts as they made their inferences, or they marked the congruent parts, using the diagram as a checklist to show that they had proven the triangles congruent (see, e.g., Fig. 2).

When P15 wrote $\overline{BD} \cong \overline{BD}$, he noted: "another thing that we can't, we can conclude, not from the given, is that B, line segment BD is congruent to line segment BD because of the reflexive, reflexive axiom." This claim and explanation communicate an understanding that some things are read from the diagram rather than the text.

For Task 7, three SPs seemed to immediately recognize how to solve the proof, so they explained a plan for the proof and marked the congruent parts immediately, prior to beginning the proof (see P20's work in Fig. 3). In effect, P20 *solved* the proof before writing out the details:

> Okay, write a proof. Given AC and DE bisect each other at B. Okay, I need to prove that AD is congruent to EC. Well I mean, again we can do congruent triangles because since they bisect each other this is going to be congruent to that and this is going to be congruent to that [marked $\overline{AB} \cong \overline{BC}$ and $\overline{DB} \cong \overline{BE}$]. And I could use the vertical angles theorem to prove that is congruent to that [marked $\angle ABC \cong \angle CBE$]. Okay, so I only need C-P-C-T-C to get the rest.

5.1.3 Students identified their warrants as postulates, axioms, definitions, or theorems

When SPs wrote or articulated warrants (i.e., reasons for their statements), they typically indicated the typology in explicit ways 86% of the time. SPs appropriately connected claims to definitions 86% of the time, sometimes even stating exact definitions (79%). For example, for Task 7, P16's explanation exhibits multiple important features:

> And then off to the side [of the flow proof], we can say that angle ABD is, and angle CBE are vertical angles. Because they branch out from point B and they're on either side. That's because of the <u>definition</u> of vertical angles. And after this, we know that angle ABD is going to be congruent to CBE because of the <u>theorem</u> we've proven: If two angles are vertical, then they are congruent.

First, P16 was "reading" the existence of vertical angles from the diagram. He then described vertical angles before stating, "That's because of the definition of vertical angles," clearly establishing that the warrant is a definition. Next, he made a new claim, stated that the warrant was a theorem, and recited the theorem.

For both proofs, the concept of congruent triangles was critical toward developing a valid proof. All SPs wrote CPCTC (i.e., Corresponding Parts of Congruent Triangles are Congruent) as the warrant for their triangle congruence statements. When asked what this meant, 100% of SPs were able to articulate what CPCTC stood for or explain what it meant. In some cases, SPs were not sure what CPCTC stood for, but they were all able to explain what it meant, as was the case for P15:

> P15: Um, C-P-C-T-C. Corresponding something, corresponding triangles are congruent.
> I…I don't remember it, but I do remember C-P-C-T-C. So.
> Interviewer: Can you tell me what it means? Just broadly?
> P15: Oh, well the corresponding sides or angles of two congruent triangles are congruent.

So even though this student was unable to say what the acronym stood for, P15 was able to articulate a reasonable definition of congruent triangles.

5.1.4 Students demonstrated that they were thinking in a logical manner

Students demonstrated that they were thinking in a logical manner as they talked through their thinking process. Specifically, as they thought aloud, SPs' explanations contained logical connectives, such as "next", "and then", and "we can conclude", in 93% of PTIs. The transcript excerpts from P18 and P14 in Sect. 5.1.1 and the excerpt from P16 in Sect. 5.1.3 are good examples of this. Logical connectives from these transcripts include: "<u>So, now I know</u> that BD is perpendicular to AC…" (P18); "<u>Next, we have</u> midpoint…" (P14); and "And <u>after this, we know</u>…" (P16). While working on Task 6, P20 used the logical connectives "then" and "so" in various ways:

> <u>So, we have</u> this is congruent to that and we have that this is perpendicular to that <u>so I guess we could use</u> the right angles theorem to prove that these are congruent <u>and then we could prove</u> that this is congruent by the reflexive property of congruence. <u>And then we can get</u> the angles congruent by C-P-C-T-C.

Although, in this more global explanation, P20 seemed to skip over the step of saying the triangles were congruent, it was included in the written proof.

5.1.5 Students attended to important details while working through their proofs

SPs attended to important details in their proofs in multiple ways. Specifically, they articulated a plan for their proof before writing it and attended to: rigor in their sub-arguments, triangle congruent criteria, and the "Prove" statement in explicit ways. SPs articulated a plan for the proof before writing the proof 64% of the time. SPs consistently attended to rigor in their sub-arguments 64% of the time. They attended to triangle congruence criteria in explicit ways 79% of the time. And they explicitly attended to the "Prove" statement 64% of the time.

In Sect. 5.1.2, the transcript from P20's work in Task 7 provides evidence of articulating a plan and attending to the Prove statement. P20 explicitly said, "I need to prove that AD is congruent to EC" and "we can do [this with] congruent triangles because" before explaining the rest of the plan. Also, in Sect. 5.1.3, P16's work in Task 7 demonstrates that he was attending to the vertical angles sub-argument by first establishing the existence of vertical angles prior to applying the theorem: if two angles are vertical angles, then they are congruent.

P16's work on Tasks 6 and 7 provides evidence of attending to sub-arguments and triangle congruence criteria. His written work is very methodical in that he established three congruent parts prior to drawing arrows in his flow proof to connect the three congruent statements to the triangle congruence statement:

> Ok so then we have our three parts [draws arrows]. So, we know that these are congruent and then we can say that triangle ABD is going to be congruent to triangle C, CBE. [Pause] I had to take a moment there to see which point was corresponding with point A. So, then we have our two triangles. And we can say this, because of S-A-S theorem. And after this, we can use my favorite theorem again to say that line segment AD is congruent to line segment EC because of C-P-C-T-C. So yeah.

The smartpen allowed us to see how the student worked out the three congruence statements prior to writing and then drawing arrows to the triangle congruence statement. In Fig. 4, we can see that the student attended to rigor in the sub-argument when he split the assumption that $\overline{BD}$ was the perpendicular bisector into two branches—one that handled the fact that $\overline{BD}$ bisected $\overline{AC}$, and one that addressed $\overline{BD}$ being perpendicular to $\overline{AC}$. From the combination of transcript and smartpen images, we can also see that he attended to triangle congruence criteria when he: said "we have our three parts," drew three arrows from the three parts before writing the triangle congruence statement, and then paused to accurately state the triangle

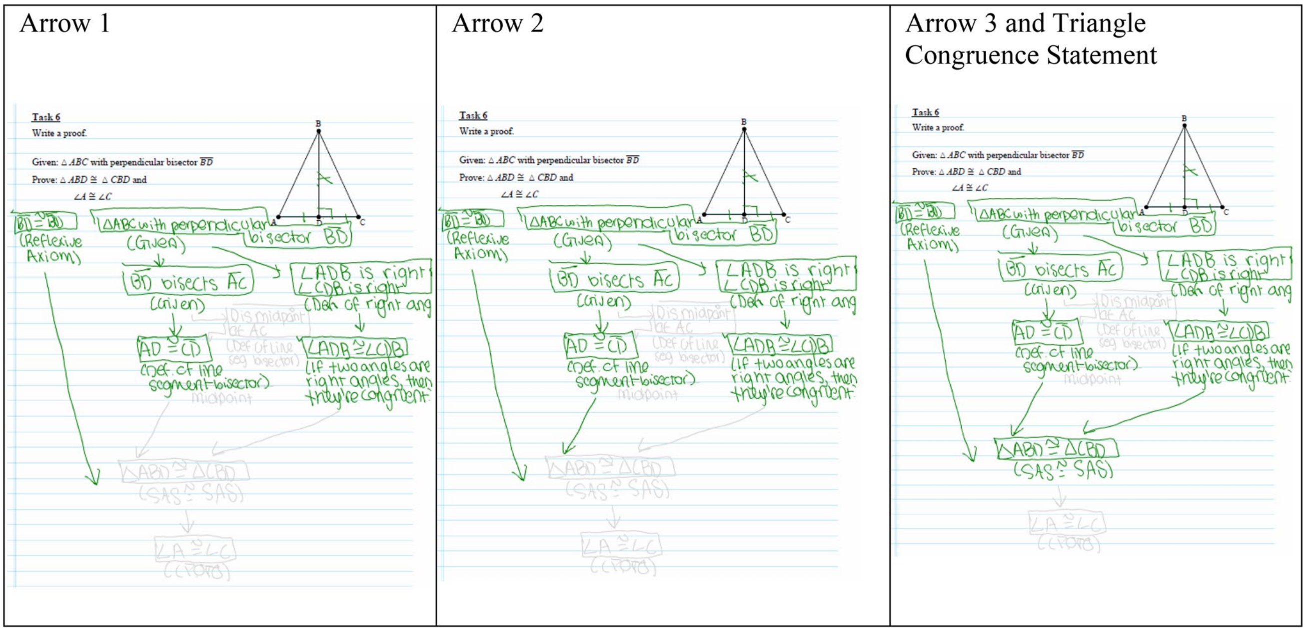


Fig. 4 P16's work on Task 6

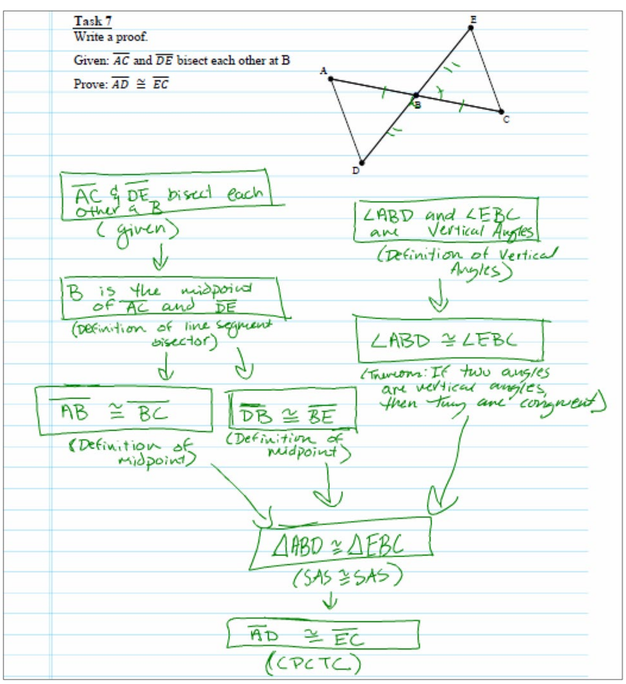


Fig. 5 P18's proof of Task 7

Table 1 SPs' use of pronouns

SP	Task	Example
15	6	"Okay, so what now **we** can conclude from that" "**You** got to write out the entire theorem to be sure"
16	6	"Okay, so then **we** have our three parts" "So, **we** know that"
15	7	"And **we** can also conclude"
20	7	"Well I mean, again, **we** can do congruent triangles"

congruence statement in a way that matched up the corresponding parts.

In P18's Task 7 work (Fig. 5), we can also see evidence of attending to sub-arguments and triangle congruence criteria. From the "Given," P18 drew one conclusion about B being the midpoint of two line segments. Yet, in order to demonstrate the three congruent parts of the triangle, she split off the congruence statements into two boxes even though she could have reasonably written them in a single box as she did for the previous step. In this way, she showed that she had three pairs of corresponding congruent parts. Also, she did not skip any important details as she wrote sub-arguments for line segment bisectors and for vertical angles. Another example of attending to details was observed when, after reading Task 7 aloud, P19 explicitly attended to the "Prove" statement as he worked through a plan for his proof:

> Alright, so that would mean that B would be the midpoint of AC and DE and that, and I can already see ahead that that probably means I'm going to have to prove this with congruent triangles cause, AD, I could see that AD and EC are corresponding.

Unlike in Task 6, where the end goal was scaffolded by asking students first to prove the triangles congruent and

Table 2 Frequencies of observed competencies for both groups (as percentages)

Observed Competencies (to nearest whole percentage)	SPs (%)	NPs (%)
Students productively attended to the "Given" information*	100	23
Students correctly identified bisectors	100	12
Students indicated what object was being bisected	93	12
Students used the diagram as a resource	100	47
Students marked the diagram	100	65
Students used the diagram as a check list or planning tool	100	46
Students made valid claims supported by assumptions about the diagram	100	31
Students identified warrants as postulates, axioms, definitions, or theorems*	86	8
Students clearly connected claims to definitions	86	12
Students stated or explained a definition	79	4
Students articulated a definition of congruent triangles	100	8
Students demonstrated that they were thinking in a logical manner*	93	8
Students attended to important details while working through their proofs	68	13
Students articulated a plan for the proof prior to writing the proof	64	15
Students consistently attended to rigor in sub-arguments	64	0
Students attended to triangle congruence criteria	79	15
Students attended to the "Prove" statement in explicit ways	64	23

*Indicates the main findings with the largest percentage gap between SPs and NPs (> 75%)

then to prove a pair of corresponding parts were congruent, Task 7 did not explicitly prompt students to prove the triangles congruent. Thus, as P19 considered a plan for the proof, he clearly kept the end goal in mind.

5.1.6 Students sometimes said *you* or *we* as they talked through their proof

The ways in which SPs used pronouns in their discourse was a noteworthy behavior (cf. competency). They used the pronouns *you* or *we* in 64% of the PTIs. Table 1 includes several examples.

It is interesting to note the ways in which some students switched pronouns as they were thinking aloud. For example, in Sect. 5.1.5, P16 used *we* three times before pausing and saying, "*I* had to take a moment there to see which point was corresponding with point A." He then resumed using *we* as he worked through his proof. Another interesting example comes from P15's work on Task 6 when he said, "So *we* got one, one angle and then well, what also *we* can conclude is that angle, what *we* should have wrote instead of this—*I* should have wrote, would be…" In both cases, SPs used the *we* pronoun, and then they stepped out of using *we* and switched to *I* as they commented on a personal point of confusion and an error, respectively.

5.2 Observations of students who were *not* successful with the proofs

In Sect. 5.1, we shared five competencies and one behavior from the SP data. In this section, we expand on our findings by considering competencies and behaviors of students who were unsuccessful with the proofs.

5.2.1 Students infrequently displayed the competencies observed in successful provers

Large discrepancies between SPs' and NPs' competencies were noted in the data. Table 2 includes frequencies of competencies from the PTIs (as percentages) for both groups for each finding and sub-finding. The differences in occurrences of the five main competencies ranged from 35 to 92% with a gap of more than 75% for 3 of the 5 main findings (as indicated in the table by *). When competencies were observed for NPs, they were similar in nature to what was described for SPs. However, as can be seen in Table 2, the percentages for NPs were typically quite low. Because there is little to say about the absence of something, for NPs, it is not so productive to go through each finding one-by-one. Instead, through three additional findings, we describe three behaviors observed in NPs' data that document what NPs did in contrast to the observations made in the SPs' data. Because the percentages of common behavioral occurrences were much lower in the NP data set, we chose a lower threshold of $\geq 20\%$ for the findings that we discuss next.

5.2.2 Students did not productively attend to the "Given" information

There were multiple issues noted in the ways NPs dealt with "Given" information. First, 23% of the time, NPs incorrectly stated the "Given" when they read it aloud. The most

Table 3 Sample of NPs' "Given" statements

"Given" for Each Task	Student	NPs' First Statement of Proof (with "Given" written as the Reason)
Task 6 Given: ΔABC with perpendicular bisector $\overline{BD}$	P2	ABC with perpendicular bisector $\overline{BD}$
	P4	ΔABD ≅ $\overline{BD}$
	P11	$\overline{BD}$ perpendicular bisects $\overline{AC}$
Task 7 Given: $\overline{AC}$ and $\overline{DE}$ bisect each other at B	P4	$\overline{AC}$ ≅ $\overline{DE}$
	P6	$\overline{AC}$ and $\overline{DE}$ ⊥ B

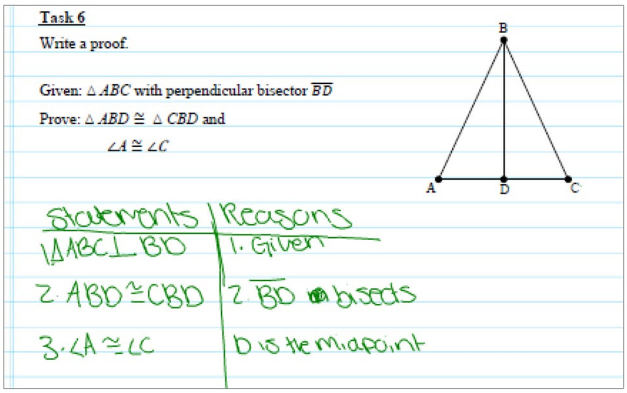


Fig. 6 P3's proof of Task 6

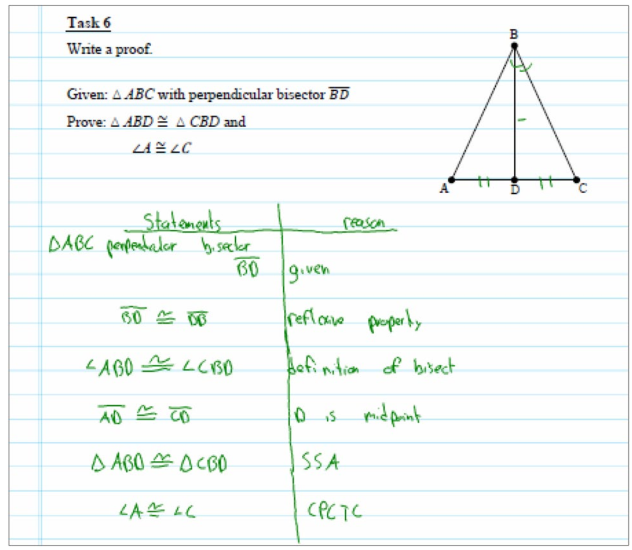


Fig. 7 P12's proof of Task 6

common error for both tasks was saying "line" rather than "line segment." Also, 46% of the time, NPs omitted notation or information when they wrote the "Given" statement in the first line of the proof (see Table 3). P11, for example, who incorrectly wrote the "Given" for Task 6, omitted it completely for Task 7. After erroneously writing the "Given" for Task 7, P4 did not continue with the proof.

NPs did not refer to the "Given" past the second line of their proof 57% of the time even though they had not exhausted all possible inferences from it. In these cases, all lines written after Lines 1 or 2 of the proof seemed disconnected from the "Given." For example, when asked about her proof of Task 6 (see Fig. 6), P3, who had not shared her thinking aloud while working, said:

> I put that triangle ABC is, bisects BD 'cause it's the Given. And ABD is congruent to CBD because line segment BD bisects the two. And I put that angle A is congruent to angle C because D is the midpoint.

There are multiple interesting things about P3's work. First, she wrote the "Given" statement incorrectly. Second, her statement in Line 2 did not identify which mathematical objects were congruent (e.g., angles or triangles) and did not seem to correspond with the assumption. Third, after "Given" in Line 1, the "Reasons" in the proof were written with respect to the diagram, rather than as general statements, as warrants should be.

Another way that NPs struggled to use "Given" information productively, was in their misidentification or vague use of mathematical objects in the "Given." It was not uncommon for students to incorrectly state ∠ABD ≅ ∠CBD in Task 6. As a reason, they tended to say or write something related to "bisect," without clearly identifying whether the bisector was a perpendicular bisector, a line segment bisector, or an angle bisector. In fact, some students seemed to interpret $\overline{BD}$ as a line of symmetry for ΔABC. Other students would simply write "Definition of bisect" as a reason for their claims (see, e.g., Fig. 7).

Finally, 39% of NPs seemed to use the diagram, rather than the "Given" to make their inferences. As discussed above, the most common occurrence of this was in Task 6. Students seemed to make claims about corresponding parts of the two triangles that looked congruent and then referenced "bisect" or wrote reasons that did not seem logical or stem from the "Given." For example, we can see that in P5's proof of Task 6 (Fig. 8), P5 marked most of the potential corresponding parts congruent and wrote five congruence statements; however, the warrants were not logical. Consequently, P5 earned 0 points on Task 6. On Task 7, a

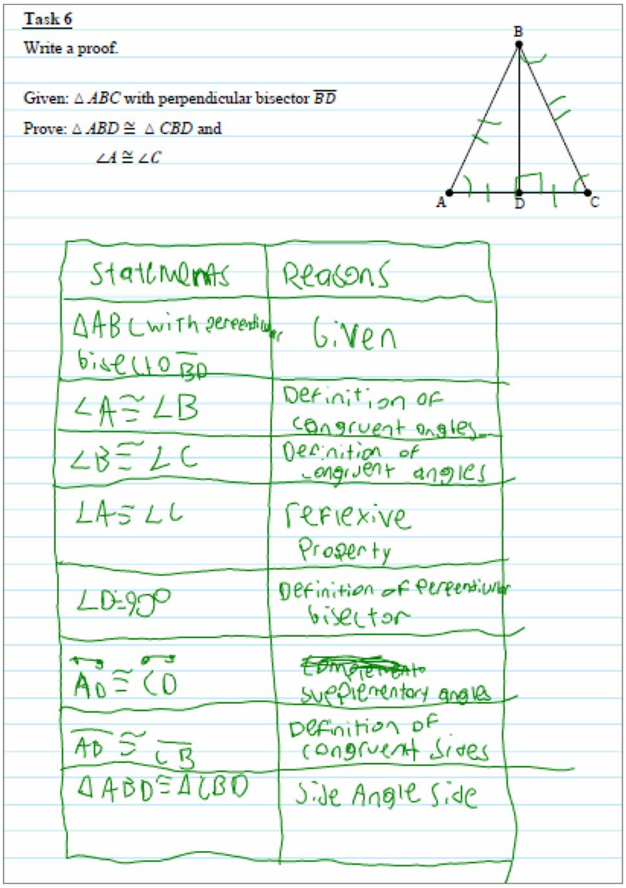


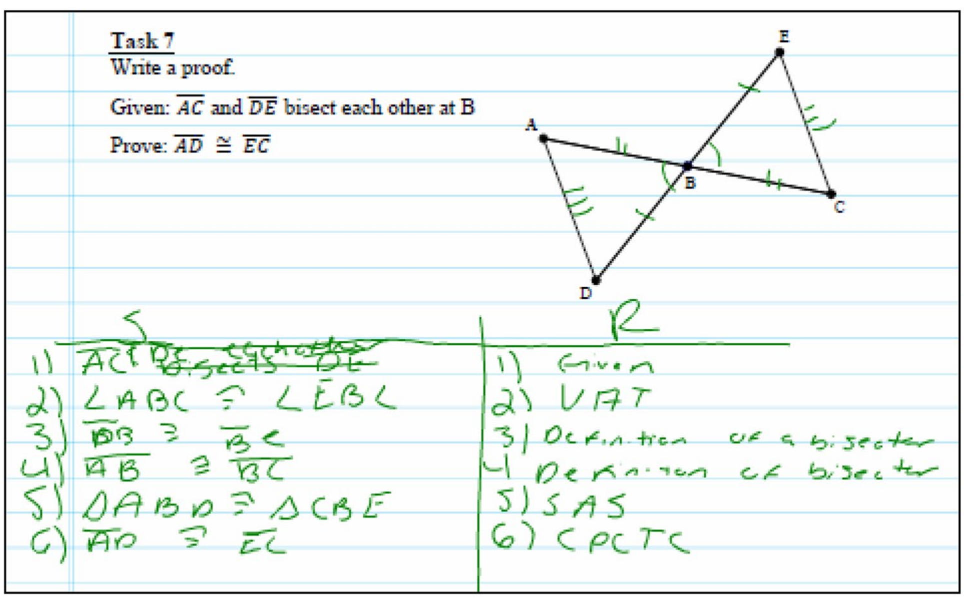
Fig. 8 P5's proof of Task 6

5.2.3 Students used vague warrants to justify their claims

NPs' warrants were vague 62% of the time. They often did not identify warrants as postulates, definitions, or theorems and generally did not seem to know definitions of relevant concepts. For example, in Fig. 7, P12 wrote "Definition of bisect" as a reason for Line 3. Yet, in order for the corresponding statement ∠ABD ≅ ∠CBD to be true, $\overline{BD}$ would have had to have been an angle bisector rather than a perpendicular bisector. Also, although the congruent segments statement in Line 4 is reasonable, the warrant references the particular diagram, rather than a definition. Similarly, in Fig. 9, P13 wrote "Definition of a bisector" as a reason for Lines 3 and 4 without identifying the type of bisector. P10′s proof of Task 6 (see Fig. 10) includes the warrant "Perp bisector" in Lines 3 and 4 without stating whether this is a definition, theorem, or something else.[8]

Another example of problematic warrants was observed with respect to the definition of congruent triangles. Only 3 of 13 NPs used CPCTC in both proofs. Two of these three students were able to articulate a definition of congruent triangles. The remaining NPs did not write CPCTC or Definition of Congruent Triangles as a warrant in either proof. Only 4 of 13 NPs wrote the final Prove statements for both proofs. Three NPs did not write the final Prove statements for either proof. When students did write the Prove statement for Task 6, but did not state CPCTC as the reason, they left the reason blank or wrote other things such as "Opp. Int. ∠" and "Reflexive Property." For Task 7, reasons supplied for

Fig. 9 P13's proof of Task 7



few students wrote congruence statements about pairs of angles that could not be justified by the assumption.

[8] Due to a technology glitch, P10's markings in this diagram are off-center but were sensibly placed on the paper.

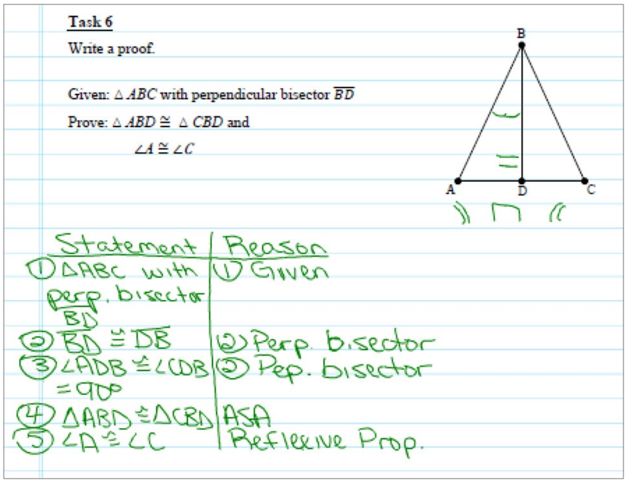


Fig. 10 P10's proof of Task 6

the final Prove statement included: "SAS," "ASS," and "B is midpoint."

5.2.4 Rather than using the pronouns *you* or *we*, NPs used *I* or did not use pronouns

In 69% of the PTIs, NPs used the pronoun *I* exclusively. Only 2 of 13 NPs used the generalizing pronouns *you* and *we*. Two students did not use any pronouns at all during the PTIs. The transcript from P3's interview in Sect. 5.2.2 is an example of a student using *I* exclusively. In one example, P5 used all three pronouns as he talked through his proof of Task 6 (see Fig. 8):

> Alright. Reflexive property. And so, *you* needed to prove that—*we* needed to prove two things. ABD is equal to CBD. So, angle D is ninety degrees. Because, wait. BD, wait. D is ninety because BD bisects AC at a straight angle.

Initially P5 used *you* and then switched to *we*. Once he became unsure, he switched to *that* and *I*. As mentioned earlier, this student earned a 0 on the proof. Last, below is an example from P11's work on Task 6:

> *I'm* also trying to remember definitions [laughs] of like angle bisector and what that definition was. Oh, *it's* a perpendicular bisector. Oh. So that would mean *it's* a perpendicular bisector. That would mean these two are right angles. Which would mean *it's* also, so that would be side-angle. Because side is congruent to itself.

Here, P11 began explaining what *I'm* trying to remember, and then she started using *it's*. In doing so, she no longer seemed connected to the proof.

6 Discussion

The data from this study provide evidence that there was a discernible set of competencies influential in students' success with triangle congruence proof tasks, and that these competencies drew on a range of cognitive skills. These findings have implications for improving the teaching and learning of proof in school geometry, and potentially for teaching proof, in general. Although some of the competencies identified here overlap with recommendations in the literature, this study shows clear correlations between the presence or absence of particular competencies and students' success or lack thereof in geometry proof problem solving. The largest differences occurred in ways in which students attended to proof assumptions, attended to warrants in their proofs, and demonstrated logical reasoning.

One way to think about problem solving is having the desire to get from the given state to the goal state, while at least initially, lacking a direct route to the goal (Mayer, 1985). Looking across the findings, we saw differences in how SPs and NPs attended to the givens and goals in the PTIs. For example, while SPs attended to the "Given" information by referring to it explicitly and making valid deductions from it 100% of the time, this competency was only observed in 23% of NP-PTIs. Instead, NPs sometimes carried out perceptual proof schemes (Harel & Sowder, 2007), seemingly drawing conclusions from the diagram, rather than the "Given" information, 39% of the time. At times, NPs incorrectly translated the "Given" statements into their proofs. In terms of the proof goal, there was a 41% gap in the frequency of attending to the "Prove" statements in explicit ways for SPs versus NPs. Related, SPs were much more likely to use the diagram as a checklist or planning tool and articulate a plan for the proof prior to writing the proof. Consistent with the findings of Koedinger and Anderson (1990) and Cai (1994), these findings indicate more of a global than local approach from SPs in the sense that SPs tended to be more focused on getting from the given state to the goal state; multiple SPs verbally stated deductions that supported moving from the "Given" to the "Prove" statements prior to writing up their proofs.

The competencies observed in the data were comprised of skills from all three cognitive domains: *knowing, applying,* and *reasoning*. Some of the largest percentage differences in competencies observed in the two groups were in the area of knowledge. For example, there was an 88% and 81% difference, respectively, in SPs' versus NPs' tendencies to correctly identify bisectors (e.g., stating that a bisector is a line segment versus angle bisector) and indicating what object was being bisected (e.g., a particular line segment versus an angle). This is important because, as noted by Herbst and Brach (2006), students' engagement with proving depends on their understanding of mathematical objects and relations involved in a problem. There was a 64% difference in the ways the two groups attended to triangle congruence criteria, which we consider a combination of *knowing* valid triangle congruence conditions and *applying* them to the particular context. There was an 85% difference in how often NPs versus SPs demonstrated that they were thinking in a logical manner, which relates to *reasoning*.

The finding related to students' identification of warrants as postulates, axioms, definitions, or theorems has an important relationship to knowledge. SPs correctly identified the typology of their warrants 86% of the time compared to only 8% of the time for NPs. Why does this matter? In Task 7, for example, students must determine, from the diagram (not the "Given"), the existence of vertical angles. This claim can be justified by the *definition* of vertical angles. A different claim about congruent angles can be made when one applies a *theorem* about vertical angles. Identifying the type of warrant, is therefore important because different claims are made when one applies the definition of a mathematical object versus a theorem about that mathematical object, of which there could be many. Also, NPs' warrants were coded as being vague 62% of the time. NPs' data also sheds light on the problem of students' "shaky knowledge" (Schoenfeld, 1985) in the area of bisectors. For example, stating as a reason, "Perp bisector," as P10 did, does not provide sufficient information about the warrant—was she referring to a theorem or the definition of perpendicular bisector, or did she not know? Also, NPs sometimes wrote, as "Reasons," particular statements about the diagram (see, e.g., Figs. 6 and 7) and did not seem aware that using warrants, written in general form, is the only valid way to justify claims.

Although we classified it as a behavior rather than a competency, we were intrigued by students' pronominal use. SPs were observed using pronouns *we* and *you* far more often than NPs. Instead, they often used *I* or *it*. Studies of pronominal use in mathematical communication have demonstrated the ways in which pronouns serve to code transactional and interactional functions of speech, such as social positioning and communication of generalization (Pimm, 1987; Rowland, 1992). For example, Pimm (1987) noted that teachers typically use *we* to represent the voice of the larger mathematical community. In contrast, Rowland (1999) posited that the use of *it* as a conceptual deictic enables students to say what they might not have been able to say otherwise. In this way, we hypothesize that perhaps NPs used *it* when they otherwise did not have the vocabulary to identify mathematical objects. Based on Pimm's observations, we hypothesize that SPs' use of *we* communicated conviction and confidence in their work which was not present in NPs' PTIs. SPs' spoken utterances tended to follow the norm of mathematicians' writing, described by Burton and Morgan (2000), in the sense that they most frequently used impersonal pronouns, while only occasionally using the pronoun *I* to state a personal preference, or, in the case of these students, to attend to an error or claim confusion.

With the exception of the findings about personal pronouns, these findings suggest that there are things we can learn about what successful provers do that contribute to the knowledge base on solving proof problems, particularly in the area of geometry. These data suggest that it would be advantageous to find ways to explicitly teach the following competencies:

- How to draw conclusions from "Given" information
- How to mark diagrams and make valid assumptions about diagrams (e.g., the existence of vertical angles; see Cirillo & Hummer, 2019)
- How to support claims with warrants, knowing the kinds of warrants that are allowable in a proof and understanding their differences
- How to logically proceed through a proof using one or more chains of reasoning, beginning with the "Given" information and ending with the "Prove" statement
- How to attend to important details in a proof, such as how to write up common sub-arguments

It is important to note that many of these competencies rely on declarative knowledge. Elsewhere, Cirillo and colleagues (2017, 2019) proposed ways in which teachers might engage students in some of these competencies in secondary classrooms, in many cases, teaching them explicitly *before* teaching proof. It is also important to note that the above list is not comprehensive; the proof tasks used for these interviews did not require students to engage in all aspects of proving, such as conjecturing or determining "Given" and "Prove" statements to prove theorems. Finally, we view the finding about pronominal use to be one of correlation rather than causation. That is, suggesting to students that they say *we* instead of *I* when sharing their thinking, will not inherently make them better at proving.

7 Conclusion

Clear differences were noted in the competencies exhibited by students who successfully solved the geometry proof problems, compared those who did not. These findings contribute to research on the teaching and learning proof in geometry, including the two topics posed in Battista's (2007) review: What components of proof are difficult for students and why? and How can proof skills best be developed in students? Identifying students' missteps and the competencies executed by successful provers during geometry proof problem solving is in important step in addressing these questions.

One recommendation for improving the field's understanding of solving proof or other problems has been to study "experts" engaged in the process (see, e.g., Iannone, 2009). A limitation of this study is that we relied on student grades to identify high-attaining students, a strategy which could be considered problematic given the potential subjectivity of assigning courses grades (cf. Weber & Leikin, 2016). Also, unlike the studies conducted by Kantowski (1977) and others who explicitly gave a knowledge test prior to executing their problem-solving protocols, we did not employ a strategy for discerning students' knowledge outside of the interview process.

The methodological choice of using smartpen technology to observe problem solving contributes to the significance of this study. The technology allowed us to "see" and analyze these data in ways that we would not have been able to see without it. For example, the active ink feature in the pencast videos allowed us to track the ways in which students marked their diagrams and shifted back and forth between diagram and proof. We conjecture that this kind of analysis is only the tip of the methodological iceberg in terms of what is possible to do with this and other tracking technologies. Future studies could, for example, track exactly how and when students mark their diagrams in coordination with their write-up of a proof. What kinds of insights into student thinking and problem solving processes might be gained from more studies like this one, is an open but exciting question.

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References

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.), *Mathematics, teachers and children* (pp. 216–235). London: Hodder & Stoughton.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Charlotte: Information Age Publishing.

Boeije, H. (2002). A purposeful approach to the constant comparative method in the analysis of qualitative interviews. *Quality and Quantity, 36*(4), 391–409.

Burton, L., & Morgan, C. (2000). Mathematicians writing. *Journal for Research in Mathematics Education, 31*(4), 429–453.

Cai, J. (1994). A protocol-analytic study of metacognition in mathematical problem solving. *Mathematics Education Research Journal, 6*(2), 166–183.

Carlson, M. P., & Bloom, I. (2005). The cyclic nature of problem solving: An emergent multidimensional problem-solving framework. *Educational Studies in Mathematics, 58*(1), 45–75.

Cirillo, M. (2018). Engaging students with non-routine geometry proof tasks. In P. Herbst, U. H. Cheah, P. R. Richard, & K. Jones (Eds.), *International perspectives on the teaching and learning of geometry in secondary schools* (pp. 283–300). Berlin: Springer.

Cirillo, M., & Herbst, P. (2011). Moving toward more authentic proof practices in geometry. *The Mathematics Educator, 21*(2), 11–33.

Cirillo, M., & Hummer, J. (2019). Addressing misconceptions in secondary geometry proof. *Mathematics Teacher, 112*(6), 410–417.

Cirillo, M., Murtha, Z., McCall, N., & Walters, S. (2017). Decomposing mathematical proof with secondary teachers. In L. West (Ed.), *Reflective and collaborative processes to improve mathematics teaching* (pp. 21–32). Reston: NCTM.

Creswell, J. W., & Poth, C. N. (2016). *Qualitative inquiry and research design: Choosing among five approaches*. Thousand Oaks: Sage publications.

Döhrmann, M., Kaiser, G., & Blömeke, S. (2012). The conceptualisation of mathematics competencies in the international teacher education study TEDS-M. *ZDM Mathematics Education, 44*(3), 325–340.

Donaldson, M. L. (1986). *Children's explanations*. Cambridge: Cambridge University Press.

Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Charlotte: Information Age Publishing.

Healy, L., & Hoyles, C. (1998). *Technical report on the nationwide survey: Justifying and proving in school mathematics*. London: University of London.

Herbst, P. G., & Brach, C. (2006). Proving and doing proofs in high school geometry classes: What is it that is going on for students? *Cognition and Instruction, 24*(1), 73–122.

Iannone, P. (2009). Concept usage in proof production: Mathematicians' perspectives. In F.-L. Lin, J.-H. Feng, G. Hanna, & M. de Villiers (Eds.), *Proof and proving in mathematics education: ICMI Study 19 conference proceedings, vol 1* (pp. 220–225). Taipei: National Taiwan Normal University.

Kantowski, M. G. (1977). Processes involved in mathematical problem solving. *Journal for Research in Mathematics Education, 8*(3), 163–180.

Koedinger, K. R., & Anderson, J. R. (1990). Abstract planning and perceptual chunks: Elements of expertise in geometry. *Cognitive Science, 14*(4), 511–550.

Mamona-Downs, J., & Downs, M. (2005). The identity of problem solving. *Journal of Mathematical Behavior, 24*(3–4), 385–401.

Mayer, R. E. (1985). Implications of cognitive psychology for instruction in mathematical problem solving. In E. A. Silver (Ed.), *Teaching and learning mathematical problem solving: Multiple research perspectives* (pp. 123–138). London: Lawrence Erlbaum Associates.

Movshovitz-Hadar, N. (2001). Proof. In L. Grinstein & S. I. Lipsey (Eds.), *Encyclopedia of mathematics education* (pp. 585–591). Abingdon: Routledge Falmer.

Mullis, I.V.S. & Martin, M.O. (Eds.) (2013). TIMSS 2015 Assessment Frameworks. Retrieved from Boston College, TIMSS & PIRLS International Study Center. http://timssandpirls.bc.edu/timss2015/frameworks.html. Accessed 18 Nov 2020.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. Abingdon: Routledge & Kegan Paul.

Reiss, K., Klieme, E., & Heinze, A. (2001). Prerequisites for the understanding of proofs in the geometry classroom. In A. D. Cockburn

& E. Nardi (Eds.), Proceedings of the 25th conference of the international group for the pscyhology of mathematics education (Vol. 4, pp. 97–104). University of East Anglia.

Reiss, K., Hellmich, F., & Reiss, M. (2002). Reasoning and proof in geometry: Prerequisites of knowledge acquisition in secondary school students. In A. D. Cockburn & E. Nardi (Eds.), Proceedings of the 26th Conference of the International Group for the pscyhology of mathematics education (Vol. 4, pp. 113–120). University of East Anglia.

Rowland, T. (1992). Pointing with pronouns. *For the Learning of Mathematics, 12*(2), 44–48.

Rowland, T. (1999). Pronouns in mathematical talk: Power, vagueness, and generalisation. *For the Learning of Mathematics, 19*(2), 19–26.

Schoenfeld, A. H. (1985). *Mathematical problem solving*. New York: Academic Press.

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning*. New York: Macmillan.

Senk, S. L. (1983). Proof-writing achievement and van Hiele levels among secondary school geometry students. (Doctoral Dissertation), University of Chicago, ProQuest Dissertations and Theses.

Senk, S. L. (1985). How well do students write geometry proofs? *The Mathematics Teacher, 78*(6), 448–456.

Sinclair, N., Pimm, D., & Skelin, M. (2012). *Developing an essential understanding of geometry for teaching mathematics in grades 9–12*. NCTM.

Sinclair, N., Cirillo, M., & de Villiers, M. (2017). The learning and teaching of geometry. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 457–488). Reston: NCTM.

Livescribe Desktop Version 2.8.3.58503. (2012). [Computer Software]. https://www.livescribe.com/site/. Accessed 4 Feb 2020.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*(1), 101–119.

Weber, K., & Leikin, R. (2016). Recent advances in research on problem solving and problem posing. In A. Gutiérrez, G. C. Leder, & P. Boero (Eds.), *The second handbook of research on the psychology of mathematics education* (pp. 353–382). Rotterdam: Sense Publishers.