Addressing MISCONC in Secondary

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EPTIONS Geometry Proof

NINE OK/GETTY IMAGES

Use these ideas to diagnose and address common conceptual obstacles that inhibit students' success.

Research suggests that teachers struggle to find effective ways to introduce proof. In 1940, in an article in this journal, Smith argued that being aware of student misconceptions in geometry is the first step in preparing to address the fundamental challenges of learning to prove. Through careful study, he identified and analyzed "three serious learning difficulties" that students have in connection with (1) a lack of familiarity with geometric figures, (2) not sensing the meaning of the if-then relationship, and (3) an inadequate understanding of the meaning of proof (p. 100). Smith found that when these difficulties were attended to explicitly, student results improved.

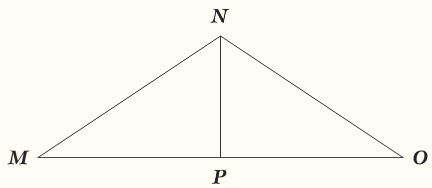
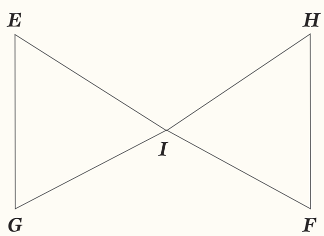
Years later, in 1985, Senk detailed findings from her study of 1520 students, in which she found that only 30 percent of students in a full-year geometry course that covered proof reached a 75 percent mastery of proof. Overall, 29 percent of the sample could not write a single valid proof. Consequently, Senk recommended (p. 455) that we must immediately look for more effective ways to teach proof in geometry, making the following suggestions:

- Pay special attention to teaching students to start a chain of reasoning.
- Place greater emphasis on the meaning of proof than we do currently.
- Teach students how, why, and when they can transform a diagram in a proof.

Many articles since have documented students' struggles with doing proof and have attempted to support teachers' efforts in teaching it. Yet we still have not had compelling empirical evidence that demonstrates new and improved methods for teaching proof in geometry. Taking a cue from Smith and Senk, we aim to identify and shed light on five common misconceptions and errors that arise when teaching proof in geometry, describe instructional methods that evoke and respond to these misconceptions, and provide sample tasks that support students and teachers to further their own understanding of these errors and misconceptions (see **table 1**). Just as Smith found in his study, we have found that student results improve when we explicitly attend to misconceptions such as those described above. In the conclusion of this article, we offer some teacher perspectives on this approach; these perspectives provide evidence that attending to particular misconceptions supports both teachers and their students and yields greater student success and enjoyment of learning proof in geometry.

We have observed these misconceptions in our experience as teachers of geometry and as researchers and teacher educators. After spending significant time in classrooms studying the challenges of teaching proof, our focus more recently has shifted to student understanding and conceptual obstacles for learning proof. Because these conceptual obstacles are fairly widespread and come early in students' study of proof, we prefer the term *conceptual obstacles* as a better and more accurate descriptor than *misconceptions*. For simplicity's sake, however,

Table 1 Five Misconceptions in Secondary Geometry Proof

Common Misconception	Sample Task to Diagnose and Remedy	Addressing the Misconception
1. You can draw conclusions from diagrams.	1. Given: $\overline{NP}$ bisects $\angle MNO$, what can you conclude from this given information? 	1. Teach students to draw valid conclusions *before* teaching proof.
2.You cannot make assumptions about diagrams.	2. What can you assume about the diagram below? 	2. Teach students what they can and cannot assume about diagrams.
3. A definition can include all the properties that one knows about the geometric object.	3. Write what you think is a good definition for parallelogram.	3. Have students practice defining, and continually emphasize the importance of knowing definitions.
4. Bisectors divide triangles in half or act as lines of symmetry.	4. In this course, we learn about three different types of bisectors. List the three types of bisectors, sketch a diagram of each one, and describe or define each type of bisector.	4. Focus repeatedly on the three types of bisectors, and formatively assess students' progress.
5. When attempting to prove a conjecture as a theorem, one assumes the conclusion of the statement.	5. Use the applet to develop and write conjectures about the sides and angles of a parallelogram. Rewrite your conjectures as conditional statements (i.e., in the "If . . . , then . . ." form). (Applet: https://tinyurl.com/PiG-Error)	5. Teach students to rewrite conjectures as conditional statements and identify the hypothesis as the "given" and the conclusion as the "prove" statement.

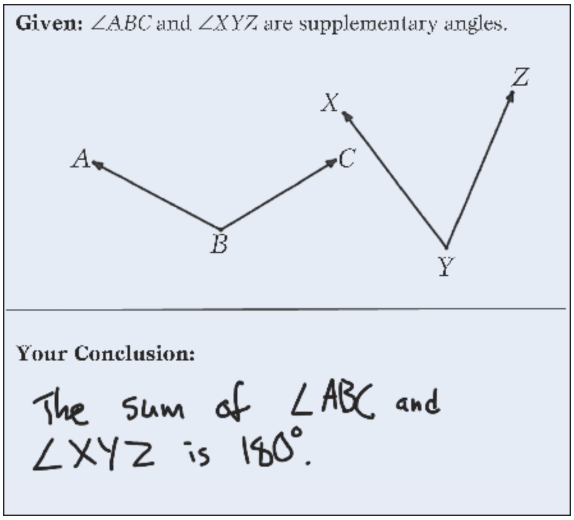


Fig. 1 After working on drawing conclusions, most SGI students concluded in the formative assessment that supplementary angles have a sum of 180 degrees.

we use the term *misconceptions* throughout the article. Misconceptions described here come from five data sources:

1. More than 150 hours of classroom observations of teaching proof in geometry
2. More than forty interviews with teachers of proof in geometry
3. Clinical interviews with twenty-nine students who earned As and Bs in their geometry proof units
4. End-of-course posttest results from an eleven-item assessment focused on proof in geometry ($n = 389$)
5. Data (written work and videos) from a two-week Summer Geometry Institute (SGI) with eleven students who were scheduled to study geometry proofs in the upcoming year

The work described here is part of a larger body of work aimed at decomposing proof in geometry (Cirillo 2014; Cirillo et al. 2017), in which we propose teaching particular proof competencies before teaching proof. Through interviews, observations, and experience, we have concluded that teachers typically use a show-and-tell approach to introduce proof. Simply stated, they show students lots of proofs and hope that students will eventually catch on. In contrast, when decomposing proof, we scaffold the introduction to proof by teaching specific competencies one at a time. As we have attempted to do this, by design, we draw out student misconceptions so that we can address them head-on and put students on a path toward valid reasoning and proving.

MISCONCEPTION 1

You can draw conclusions from diagrams.

Even though we tell students that they cannot draw conclusions from diagrams, they still tend to draw conclusions on the basis of what "looks" true, rather than on the given information and relevant definitions. This is especially so when first learning proof. For example, when asked to solve the first task in **table 1**, before learning proof, most students draw conclusions about the entire figure, rather than just stating that $\angle MNP \cong \angle ONP$. On the end-of-course posttest, when asked a question similar to task 1, only 28 percent of students ($n = 389$) correctly concluded that the only valid conclusion was that two angles were congruent ($\angle MNP \cong \angle ONP$). More than half of the students assessed (56 percent) believed that all the following were also true: $\overline{NP} \perp \overline{MO}$, $\overline{MP} \cong \overline{OP}$, $\angle M \cong \angle O$, and MNO is an isosceles triangle.

So, even when students have already studied proof, without some sort of intervention, most students do not answer this task correctly, including undergraduate math majors.

Because conceptual obstacles are fairly widespread, we prefer this term over *misconceptions*.

ADDRESSING MISCONCEPTION 1

Teach students to draw valid conclusions before teaching proof.

One strategy for addressing this common misconception is to teach students explicitly to draw conclusions. Drawing valid conclusions is a competency that we can teach in isolation, separate from and as a prerequisite to teaching proof. In doing so, we address the misconception that it is reasonable to draw conclusions on the basis of how the diagram looks, and we can teach students how to draw valid conclusions using the given information and relevant definitions. On the basis of the feedback we have received from classroom teachers, we conclude that the Drawing Conclusions tasks are the single most influential change in practice that can be made when teaching geometry proof. Formative assessment items given to students after learning to draw conclusions indicate that, with the exception of using correct notation (e.g., $m\angle ABC$), this approach was effective in supporting students to draw valid conclusions (see **fig. 1**).

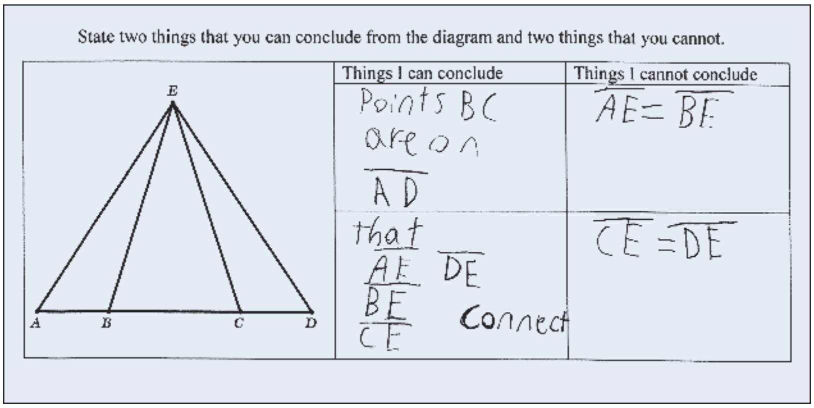


Fig. 2 After teaching the assumptions in **table 2**, students completed this formative assessment.

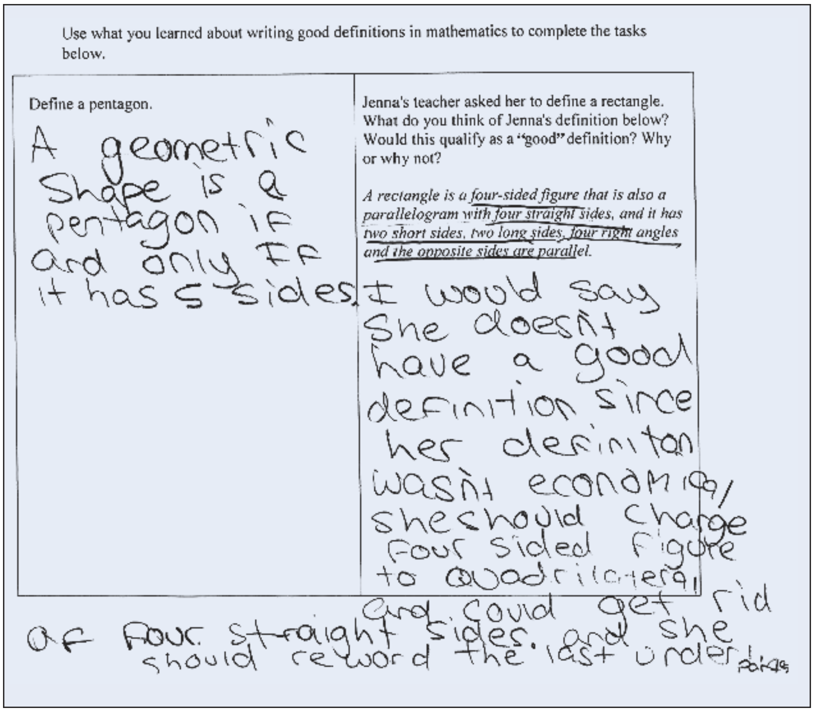


Fig. 3 An SGI student defines pentagon and critiques a student definition as part of a formative assessment of the defining lesson.

Table 2 A Sample of Assumptions about Diagrams

What You *May* Assume	What You *May Not* Assume
• If two straight lines intersect, they intersect at one point.	• If lines look parallel or perpendicular, they may not be.
• If points look collinear, they are.	• If angles or segments look congruent, they may not be.
• Relative location of points—if a point looks like it is between two other points or to the left or right of a point, it is.	• Relative size of segments and angles—if one angle (or line segment) looks bigger than another, it may not be.
• If angles look adjacent, they are.	• If angles look like right angles, they may not be.
• If angles look like vertical angles, they really are.	

MISCONCEPTION 2

You cannot make assumptions about diagrams.

The flip side of the not-drawing-conclusions-from-diagrams coin is that students come to believe we cannot make *any* assumptions about diagrams. Given that students have heard teachers make such remarks as, "We can't say something is true just because it looks like it's true," this is a reasonable conclusion. We have said this ourselves; however, it is false. Indeed, there are assumptions that we can, and, in fact, we *must* infer from diagrams. Some textbooks actually do a good job of laying these out explicitly. However, we and other teachers we observed and partnered with never explicitly addressed this idea with students.

ADDRESSING MISCONCEPTION 2

Teach students what they can and cannot assume about diagrams.

Diagrams play a critical role in the construction of meaning in geometry. For this reason, we have found it important to be explicit about what assumptions we can make about diagrams, and these things should be taught in contrast to those that we cannot assume (see **table 2**). For example, we can make assumptions about the affine nature of geometric objects, such as betweenness or the relative location and collinearity of points, but we cannot make assumptions related to measurement, such as congruence or relative sizes of segments and angles. A review of textbooks indicates that if this idea is addressed at all, only the assumptions one can make are addressed. However, distinguishing these from those assumptions that one cannot make essentially helps to establish the ground rules for working with diagrams, and it supports students in drawing conclusions. After teaching this competency explicitly, using a task similar to task 2 in **table 1**, we found that, with the exception of still needing to learn proper notation, most SGI students developed this competency (see **fig. 2**).

MISCONCEPTION 3

A definition can include all of the properties that one knows about the geometric object.

Students have interesting ideas about mathematical definitions. Because they tend to come to geometry with many concept images of the geometric objects being studied, they often have partial ideas about definitions of these objects. Additionally, they tend to know many properties of these objects, including those that are not included in their definitions. Therefore, when asked for definitions of terms like *isosceles triangle* or *parallelogram*, students have a tendency to state everything they know about the concepts. For example, when asked to write a definition of parallelogram, students write about shapes that have opposite parallel and congruent sides and opposite angles that are congruent, rather than stating only the necessary and sufficient information.

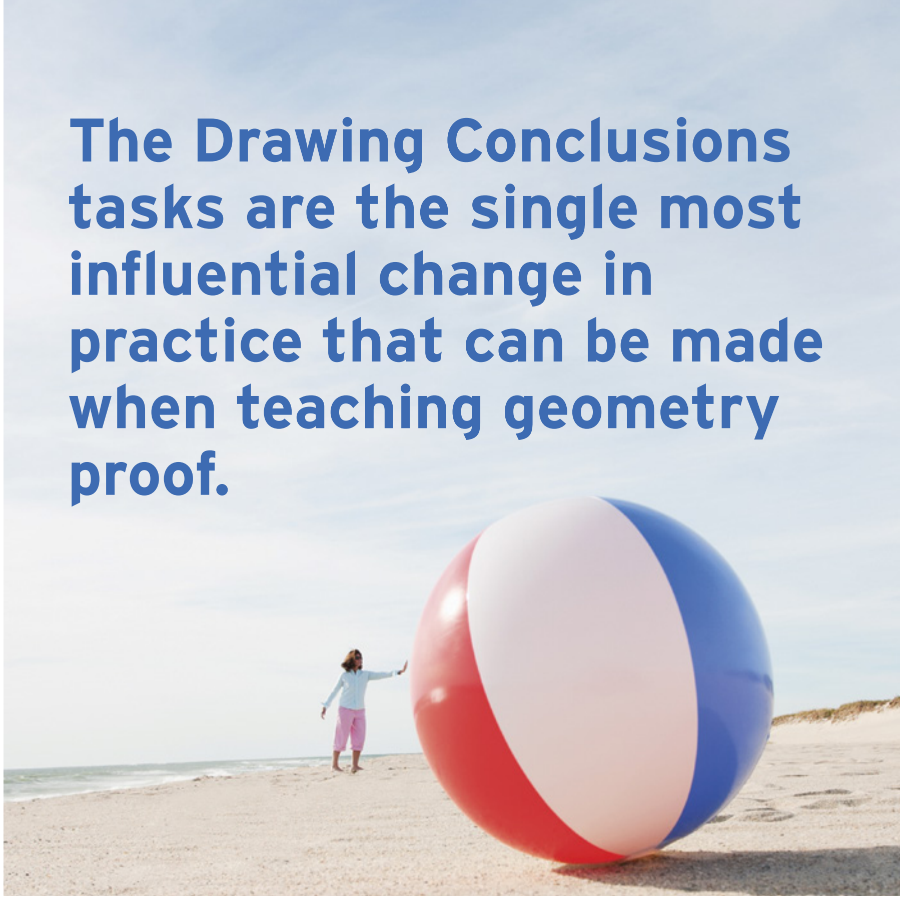
ADDRESSING MISCONCEPTION 3
Have students practice defining, and continually emphasize the importance of knowing definitions.

Engaging students in defining develops their appreciation of what a "good definition" consists of and supports their attention to precision. For example, students can appreciate that a good definition should state "what it is" and "what makes it special." That is, when defining, identifying the class of objects that a geometric object falls within is important (e.g., "An isosceles triangle is a triangle . . ." or "A parallelogram is a quadrilateral . . ."). As well, stating what is special about that particular class of objects is essential (e.g., ". . . that has two congruent sides" or ". . . that has two pairs of opposite parallel sides"). Additionally, because we want students to understand that definitions are "reversible," we suggest that they rewrite their definitions as biconditional statements to emphasize that definitions, unlike theorems, are always true in both directions (e.g., A triangle is an isosceles triangle if and only if it has two congruent sides).

In our experience, having students define *all* important terms is unnecessary and impractical; however, asking students to engage in defining does decrease student comments about teachers being "too picky" about language. In particular, through their defining efforts, which should include critiquing and revising others' definitions, students develop an appreciation for providing just enough but not too much information in their definitions. Tasks such as task 3 in **table 1** and **fig. 3** can support students while they develop and critique definitions.

MISCONCEPTION 4
Bisectors divide triangles in half or act as lines of symmetry.

Most teachers of secondary geometry know that when it comes to line segment bisectors, students have a difficult time identifying which line segment is being bisected. We observed this over the years as we taught proof ourselves. Not until we really observed and analyzed student thinking, however, did it became clear just how challenging it is for students to comprehend the different bisectors used in geometry proofs. That is, students are challenged to draw valid conclusions about angle bisectors, line segment bisectors, and perpendicular



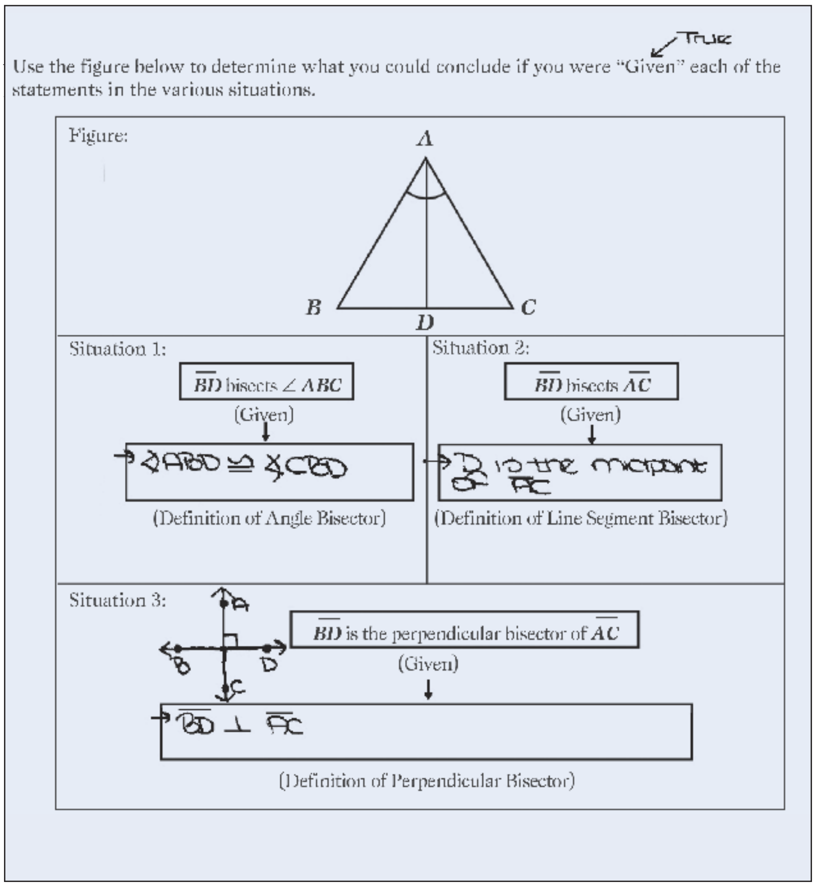


Fig. 4 This SGI student work sample reflects a typical response to this formative assessment. Most students' answers, as is the case here, were correct for the first two situations but incomplete for Situation 3.

conclusions, however, we found errors about bisectors particularly challenging to correct.

ADDRESSING MISCONCEPTION 4

Focus on the three types of bisectors repeatedly, and formatively assess students' progress.

Through our research, we found that dealing with misconceptions around bisectors could not be fixed in one Drawing Conclusions lesson. Drawing Conclusions tasks are a good way to begin to address and diagnose these misconceptions, but we have found the need to attend to this issue repeatedly across many lessons. When confronted with a "given" statement about a bisector, it is always useful to ask, "What is being bisected?" or "What type of bisector is this?" Then follow up with reminders of a definition of that particular bisector. During the SGI, we found the need to formatively assess students' developing understandings of bisectors repeatedly to determine whether they could distinguish them (see **fig. 4**). Pointing out that a perpendicular bisector is a special kind of line segment bisector is also important. Teaching these differences in advance and outside of the proving process is likely to be more effective than teaching them while students are learning to prove.

MISCONCEPTION 5

When attempting to prove a conjecture as a theorem, one assumes the conclusion of the statement.

When students are asked to write conjectures, naturally, they tend not to write them as conditional statements. For example, when asked to develop conjectures using technology, as in task 5 in **table 1**, students typically write, "Opposite sides [or angles] are congruent." Even when provided with a conjecture, students have difficulties rewriting the conjecture as a conditional statement. For example, when 389 students were asked to rewrite the conjecture, "The diagonals of a parallelogram bisect each other," as an "If . . ., then . . ." statement, only 14 percent of students earned full credit on this item. A common error was swapping the hypothesis and the conclusion. Similarly, when students were asked to determine what was given and what they would be trying to prove for the conjecture "The diagonals of a rectangle are congruent," (see **fig. 5**), of the fourteen students interviewed who earned As and Bs in the proof unit, ten students did not write the statement $\overline{AC} \cong \overline{BD}$ anywhere in the "Given" or "Prove" statements. These data show two things. First, students have difficulties converting their conjectures into conditional statements. Second, students have difficulties selecting what they are assuming to be true and what they will be trying to prove from conjectures written as "regular" sentences and then applying them in particular figures.

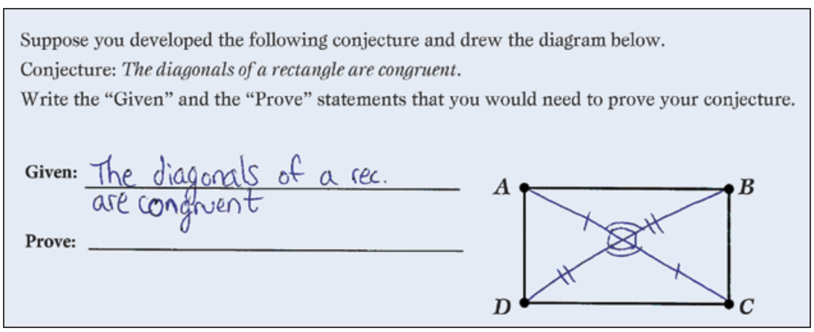


Fig. 5 Thirteen of fourteen students interviewed wrote the incorrect "Given" statement after earning As or Bs in the geometry proof unit.

Students should be given opportunities to develop their own conjectures, not just write proofs of statements provided to them.

ADDRESSING MISCONCEPTION 5

Teach students to rewrite conjectures as conditional statements and identify the hypothesis as the "Given" and the conclusion as the "Prove" statement.

To support students' understanding of what it means to prove something mathematically, students should be given opportunities to develop their own conjectures, not just write proofs of statements provided to them. To address the challenges of doing so, we suggest allowing students to write their conjectures in any form and then asking them to rewrite their conjectures as conditional statements. We will likely need to guide students in this rewriting. From there, we can teach them to identify the hypothesis and the conclusion of the conditional statement, and then we need to support students in applying these to particular figures. Keeping in mind that current standards suggest that students should be proving geometric theorems, and because constructing viable arguments and critiquing the reasoning of others include making and exploring conjectures, addressing this misconception is important.

GREATER SUCCESS, LESS RESISTANCE

We wholeheartedly agree with Smith, who argued more than seventy-five years ago that being aware of student misconceptions is the first step in preparing a successful geometry course. We outlined specific misconceptions or conceptual obstacles that we found through our own work, and we offered suggestions for addressing them. Classroom teachers with whom we have worked have taken these suggestions and found greater success with teaching proof. Using this approach, teachers have found less student resistance to learning proof and have even found that many students developed positive dispositions toward learning proof. In conclusion, we give the teachers with whom we have worked the last lines of this article:

- "I thought that teaching students to draw conclusions before teaching proof itself was one of the most useful things that I learned in this professional development."
- "Having the students write definitions helped because, this year, they weren't asking me in the middle of a proof, 'Why can I say that?' This time, they knew that they probably had a definition."
- "Students were also able to work with each other to fix each other's proofs. I am able to walk around and help students, but I don't have to hold their hands through every step. I also don't have to help students start a proof. At the very least, they mark the diagram, write the given, and try to draw a conclusion or two."

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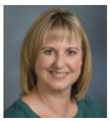

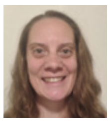

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Let's Chat about Misconceptions

On Wednesday, April 24, at 9:00 p.m. ET,

we will discuss "Addressing Misconceptions in Secondary Geometry Proof," by Michelle Cirillo and Jenifer Hummer (pp. 410–17).

Join the discussion at #MTchat.