# EXTRACTIVE AND INFERENTIAL DISCOURSES FOR EQUATION SOLVING


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*We investigate the algebraic discourse of secondary mathematics teachers with respect to the topic of equation solving by analyzing five teachers' responses to open-ended items on a questionnaire that asks respondents to analyze hypothetical student work related to equation solving and explain related concepts. We use tools from commognitive analysis to describe features of teachers' explanations, and we use these survey responses as examples to illustrate a distinction in discourses about equation solving that has implications for students' learning of common procedures for finding solution sets of equations and systems.*



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The Common Core State Standards suggest that students should come to view equation solving as a form of inquiry whose goal is to identify all solutions of an equation, and learn that steps in an equation-solving process represent successive deductions about a hypothesized solution (6.EE.5, A-REI.1, NGA & CCSSO, 2010). However, discourse about equation solving in algebra courses does not always capture this sense of discovery and deduction (Patterson & Farmer, 2018). In this report we investigate inservice teacher thinking about solving equations with respect to its treatment of mathematical objects, symbols, and routines and explore implications for classroom discourse and opportunities for students' algebraic reasoning.

## Theoretical Framework

In describing discourse about equation solving, we mark a distinction between an *extractive discourse* and an *inferential discourse* for solving equations. In the extractive discourse, one describes an equation-solving routine as a sequence of actions on mathematical symbols. In the inferential discourse, one describes an equation-solving routine as using properties of numbers and relations to generate a sequence of endorsed narratives (or inferences) about a hypothesized solution to an equation or system. The routine as a whole produces a conditional: "If the original equation [or system] is true, then the value of the variable must be …" One defining distinction between extractive and inferential discourses is that extractive discourse contains more lexical markers of human agency in the equation-solving process, such as "I moved the $2x$ to the other side" or "you need to set both factors equal to zero." This is consistent with extractive discourse as primarily focused on actions on mediators ("moved", "set", "plugging") and is an example of *personalization* in mathematical discourse (Ben-Yehuda, Lavy, Linchevski, & Sfard, 2005).

Our focus on the concepts of extractive and inferential discourse is rooted in the work of Sfard (2016) who described *ritualized* and *explorative* participation in mathematical discourse. Ritualized discourse is defined as a "discourse-for-others" (Sfard, 2006) in which learners talk about mathematics according to the goals and motivations of others, without clearly identifying mathematical objects (such as numbers, functions, or solutions) as objects of the discourse. By contrast, in explorative discourse, participants strive to know more about mathematical objects and are not constrained to reasoning moves and routines set by others. Literature from the commognitive perspective indicates that explorative participation in mathematical discourse entails meaning for the objects of the discourse that may not be accessible to learners engaged in ritualized participation (e.g., Ben-Yehuda et al., 2005). We hypothesize that developing an inferential discourse for equation solving entails developing three mathematical meanings (Thompson, 2013): (1) to solve an equation is to find value(s) of the variable(s) that make the equation true; (2) we can assume that the variable(s) have value(s) that make the equation true, and each step in the process asserts that an equality is true provided that the previous equality is true; and (3) the converses of the conditional statements generated in this process may or may not be true, depending on whether the functions applied to both sides are invertible.

As part of a larger study (NSF Award #1908825), we use these ways of thinking about teacher mathematical discourse to investigate the question: *What language do middle and high school mathematics teachers use to describe and explain routines commonly used in algebra?*

## Method

A 13-item survey on algebra concepts was administered to five teachers. This report focuses on responses to two items on procedures for solving equations (Table 1). The participants are teachers in an urban school district in the southern United States. Diann, Felicia, and Teodora are high school teachers, while Vanessa and Tanya are middle school teachers. All teachers were teaching at least one Algebra 1 class at the time they completed the survey. We analyzed each teacher's responses, noting how their uses of words and mediators, endorsements of narratives, and descriptions of routines aligned with extractive or inferential discourses for equation solving.

**Table 1: Two Items in the Solving Equations Strand**

| **Description / Questions Asked (Separate cells indicate separate pages)** |
|---|
| **Meaning of Solve:** "What does it mean to *solve an equation*?" |
| [A correct solution of the equation $13 + 3x = 48 - 4x$ is provided.]<br>"Thinking about this problem-solving process as a whole – without analyzing each individual step – why does this process produce a number ($x = 5$) that is a solution to the original equation given?" |
| **Special Systems of Linear Equations**<br>[A correct solution of the inconsistent system of equations $\{15x + 3y = 33; 5x + y = 14\}$ using the substitution $y = 14 - 5x$ is provided. The hypothetical student obtains the equation "42 = 33" and writes, "This is never true, so the system has no solutions."]<br>In the work above, is the equation $15x + 3(14 - 5x) = 33$ a true statement? Why or why not?<br>The student then simplifies the equation $15x + 3(14 - 5x) = 33$ to obtain the equation $42 = 33$. Is this new equation a true statement? Why or why not?<br>Does the reasoning shown here support the conclusion that the system has no solutions? If so, explain why. If not, explain what is wrong with the reasoning shown. |
| [A solution of the dependent system of equations $\{4x - 12y = 28; x - 3y = 7\}$ using the substitution $x = 3y + 7$ is provided. The hypothetical student obtains the equation "28 = 28" and writes, "This is true for all numbers $x$ and $y$, so <u>all</u> ordered pairs $(x, y)$ are solutions."]<br>Does the reasoning shown here support the conclusion that all ordered pairs $(x, y)$ are solutions to the system? If so, explain why. If not, explain what is wrong with the reasoning shown. |

## Results and Analysis

In our analysis we describe teachers' responses to the three items in Table 1, whose names we abbreviate *Meaning* and *Systems*. Our goal is not to characterize any one participant's discourse for equation solving as particularly extractive or inferential; we found that each teacher's responses contained elements of both, often within the same response.

**Discourse About the Meaning of "Solve" and the Equation-Solving Process**

Each teacher's response to the first page of *Meaning* suggested that solving an equation involves finding a value (or all values) of the variable that make the equation true. Responses differed with respect to the importance of finding all solutions of an equation: Felicia said, "Solving an equation is usually finding the value(s) of a variable that makes that equation true," while Teodora said, "Solving an equation means finding a value of $x$, that when substituted, will make the left side of the equation equal to the right." We do not consider this distinction as having any bearing on the extractive-inferential distinction. However, we also note that while Felicia describes solutions as "the value(s) of a variable that makes that equation true," suggesting independence from the actions of the solver, Teodora's description is suggestive of the process of substituting a value for the variable to verify that it is a solution. We therefore characterize Felicia's response to this question as closer to the inferential end of the spectrum because it describes the idea of solution in a manner independent of human action.

On the second page, Tanya's and Teodora's responses only verified that 5 is a solution of the original equation. By contrast, Diann and Felicia addressed the equation-solving process directly. Diann said, "By performing the inverse operations on both sides of the equation, you are reversing the operations on the $x = 5$ that ended with that result." This response focuses on performing appropriate actions on the "sides of the equation" based on the structure of each side; the word "reversing" may refer to the order in which these actions should be taken. This focus on actions on signifiers points toward extractive discourse. Felicia responded, "There is an assumption that both sides are equal and basically the whole process is manipulating things while keeping that equality until the $x$ is isolated." This response also contains markers of extractive discourse ("manipulating things," identifying the step when the mediator $x$ is "isolated" as a termination condition for the routine), but also stipulates the assumption that the two sides are equal and states that equality should be preserved at each step, which points toward inferential discourse. Both discourses have benefits to offer: the inferential discourse focuses on the equality of values of the expressions at each step, while the extractive discourse highlights strategic knowledge that would help a person reduce the equation to a simpler form.

**Shedding Light on Special Cases: Discourse About Linear Systems**

The item *Systems* asks respondents to address implications that occur in the process of solving a "special" system of two linear equations in two variables. The first question on the first page asks whether the equation $15x + 3(14 – 5x) = 33$, resulting from a correct substitution, is a true statement. The next question then asks whether the equation $42 = 33$, a correct simplification of the prior equation, is a true statement. From an inferential perspective, the equation $15x + 3(14 – 5x) = 33$ is true under the assumption that $(x, y)$ is a solution, and the fact that this statement implies the false statement $42 = 33$, yielding a contradiction, shows that our original assumption (that there is a solution) must be false. We analyzed how teachers dealt with the apparent contradiction in saying that the equation $15x + 3(14 – 5x) = 33$ is true but $42 = 33$ is false.

Teodora considered the first question from a global perspective: "No, it is not a true statement. When solving, we are looking for a value that will make it true. The student finds that solution does not exist and therefore the statement will never be true." This points toward inferential discourse: because the truth of the equation $15x + 3(14 – 5x) = 33$ (along with the truth of the equation $y = 14 – 5x$, not mentioned in the response) would imply the existence of a solution, and we know that no solution exists, it is not possible for the equation to be true. On the other hand, Tanya, Diann, and Felicia all responded that this equation was true. Tanya said, "Yes, this used substitution. Substitution is one method for solving a system of equations." This response endorses the narrative $15x + 3(14 – 5x) = 33$ based on the use of a standard routine without reference to properties of numbers and relations, and thus we view it as an example of extractive discourse. Diann's and Felicia's responses were both suggestive of taking the truth of the original equation $5x + y = 14$ or the equivalent $y = 14 – 5x$ as a premise; Diann said, "From the above line if $y = 14 – 5x$ then $y$ is equal to $14 – 5x$ so they can be replaced to represent the same value. It did not change the value of the equation since they were equal before the substitution." While this response contains some references to actions on symbols ("replaced"), it also grounds its argument in narratives about the equality of numbers based on the assumption that an original equation is true, which we expect to find in inferential discourse.

Asked about the truth of $42 = 33$, Diann responded, "The simplification was correct. After distributing the 3, the terms with $x$ will cancel out to zero. The new equation $42 = 33$ is not a true statement because those two numbers are not equal or the same." Felicia said, "The new equation is not a true statement, $42 \neq 33$." We wondered how each teacher viewed the conclusion that the system has no solutions. Diann said simply, "The reasoning is correct, this system has not [*sic*] solution. There is no coordinate pair $(x, y)$ that will make the equations true." Felicia said, "The reasoning is that there is no value of $x$ that will give us a true statement, therefore no solution to the system." While we cannot be certain what Felicia meant by "true statement," the fact that she identified the statement $42 = 33$ as untrue suggests that she understands that no value of $x$ will avoid this contradiction and concludes that no value of $x$ (and $y$) can solve the system.

## Discussion

We view extractive and inferential discourses as complementary ways of communicating about processes for solving equations. While the extractive discourse provides access to language and narratives that help solve equations mechanically and fluently, the inferential discourse offers a conceptual microscope under which learners can examine unexpected wrinkles in solution processes. Developing an inferential discourse for equation solving may unlock opportunities for productive struggle in students' learning of algebra, because this discourse allows learners to examine novel features of equation-solving processes based on foundational principles rather than uncritically memorizing routines for classes of problems. Investigating teachers' discourse about equation solving is an important first step in this work because their discourse can afford or constrain students' opportunities for conceptual thinking.

Our analysis is based on teachers' untimed responses to survey questions. Because teachers understood that they were explaining concepts for researchers and not for their own students, we cannot claim these survey responses as a model for explanations that teachers might give in the algebra classroom, where timing, assessment of students' needs, and curricular context might influence decisions about discourse. However, the variety of responses to the two items in this report is evidence of the diversity of explanations available to teachers when they encounter an opportunity for conceptual development. While teachers may not have access to all of these explanations depending on their knowledge and prior experience, we anticipate that through collaboration and professional development teachers may gain access to a greater range of discursive tools for helping students build conceptual understanding of algebraic procedures.

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