WHAT IS "REPEATED REASONING" IN MP 8?

The Common Core State Standards (CCSSI 2010) for Mathematical Practice have relevance even for those of us not in CCSS states because they describe the habits of mind that mathematicians—professionals as well as proficient school-age learners—use when doing mathematics. They give us a language to discuss aspects of mathematical practice that are of value in all endeavors that require analytic thinking. Unlike the content standards, which are tightly targeted to specific grades, the practice standards are intended to apply across all content areas and at all grades, acquiring greater depth and broader application over the years.

For studying new problems or building new theories, mathematicians often use the practice of trying concrete examples, looking for regularity in the reasoning, and describing the pattern in the reasoning. The CCSS describes this practice, MP 8, this way:

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> Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1) \bullet (x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results. (CCSSI 2010, p. 8)

To make sense of MP 8, we illustrate what "repeated reasoning" means, why looking for and expressing regularity in it is such a valuable mathematical habit of mind, and how that differs from analyzing structure (MP 7) and from finding patterns in numerical results.

NOT ALL PATTERN FINDING IS MP 8

Looking for regularity in reasoning is different from looking for patterns in numerical results. Both are valuable skills and both can sometimes be needed in the same problem, but they are different. Consider the problem shown in **figure 1**, a type often given in algebra classes.

A typical student approach, often heavily scaffolded in the way the problem is posed, is to build several small rows of triangles, create a table, look for a numerical pattern, and then try to find a linear function that matches that pattern. There is repetition: building rows. There is pattern-finding: scanning the numbers in the table to look for relationships. Neither necessarily exposes pattern or regularity in the reasoning.

LOOKING FOR REGULARITY IN REPEATED REASONING: TWO EXAMPLES

We now describe two student approaches that illustrate looking for regularity in repeated reasoning. The problems are not special; the first neither requires nor especially pulls for MP 8. Rather, the students' *approaches* to the problems illustrate this mathematical practice.

Creating an Equation to Solve

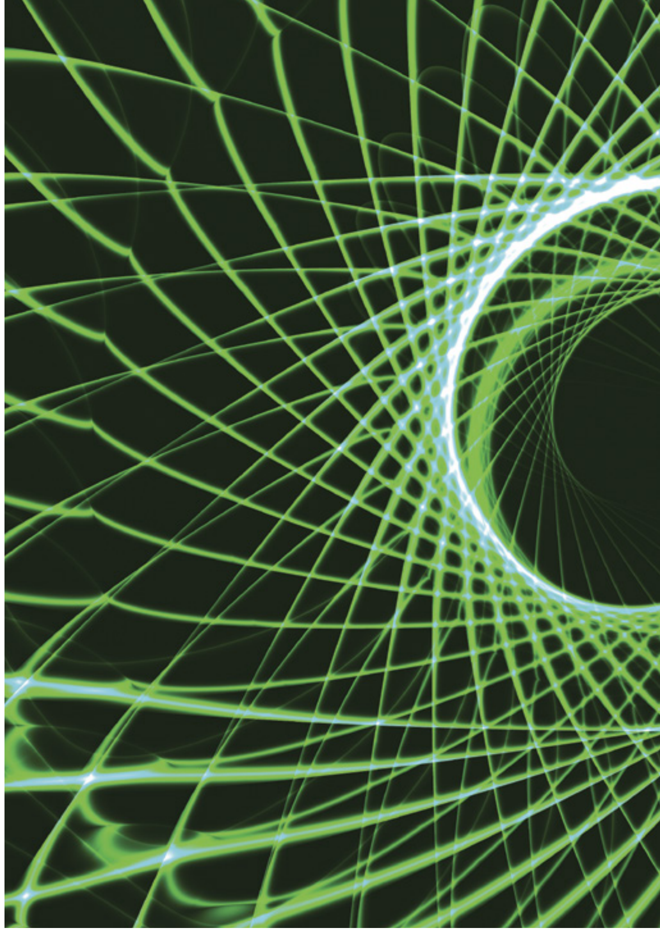
> Tamara was working half time, 20 hours per week. She got a raise of \$3.50 per hour and also started working full time, 40 hours per week. Now she earns \$370 more per week than she did before. Write and solve an equation to determine how much she earns per hour now.

Algebra students benefit from the learned habit of looking for and expressing regularity in repeated reasoning.

In this problem, the student is expected to create an equation and solve it. CCSS differs from earlier standards by distinguishing those two steps: the creation of expressions, functions, and equations to model a situation, and the manipulation and use of those objects. For many students, creating an equation is harder than solving it. Finding regularity in repeated reasoning can help.

Mathematicians often approach problems for which they do not already have a formula or a solution method by trying concrete cases. The aim is not to find the numerical (or other) answer by chance or approximation, nor is it to practice the calculations. The aim is to come to understand the flow of the calculations—their nature and organization, their regularity—well enough to invent a general formula or solution method.

Sarah, in her first year of algebra, uses that approach in the Tamara problem, as shown in **figure 2**. She picks an easy-to-use number for the new wage, figures out the old wage, and uses those numbers to compute what Tamara makes now and what she made before. Finally, Sarah subtracts to see if the result is $370. It isn't. She notes that with a little sad face, picks a different easy-to-use number, and tries the calculation again.



Here are two rows of triangles made of toothpicks. One has 3 triangles; one has 6. Write an expression that says how many toothpicks are needed for a row of n triangles.

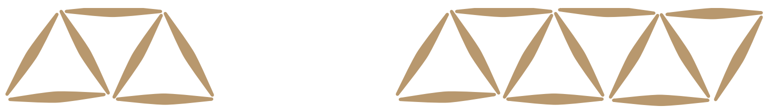


Fig. 1 Problems like the Matchstick problem are often given in algebra classes.

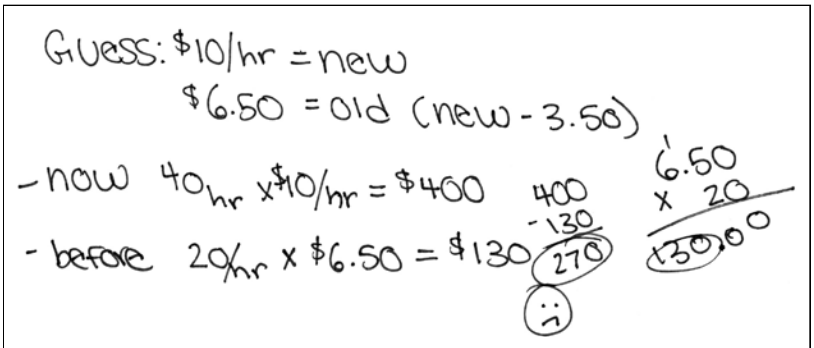


Fig. 2 Sarah's first trial calculations show that the difference in earnings is $275.

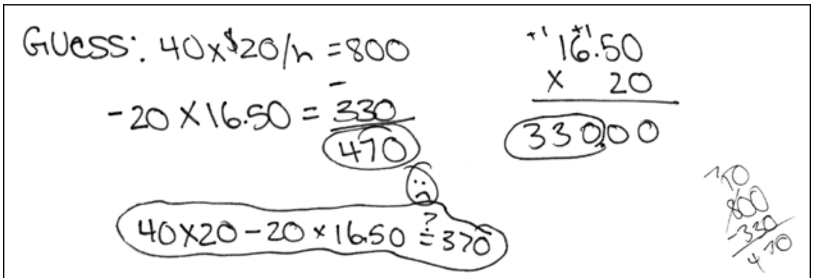
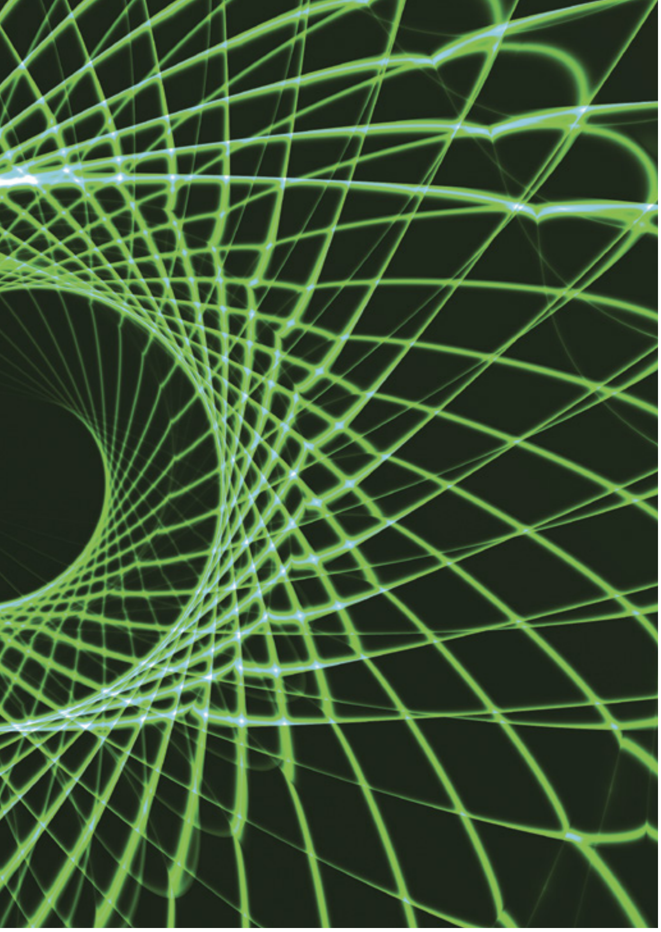


Fig. 3 Sarah's second trial calculations still do not yield the desired $370 difference.

As she did before, she computes the old wage and both the new and old earnings, then subtracts to see if it is $370. (See **fig. 3**.) Again it isn't; but even if it were, she isn't done: Her goal is still the equation. That is the purpose of the experiments.

So she tries again. Until students are proficient at this process, they may need many trials before they begin to distinguish all of the changed numbers from the aspects of the computation that do not change—the repeated reasoning; but even for accomplished mathematicians, the repeating aspects of a computation become apparent only because they repeat. This takes experimentation. Sarah's particularly clear records make the thinking behind her experiments easy to see. The neatness is partly just Sarah, but recording each experiment in a discrete location, in a way that can be re-read and that exposes the reasoning, is something that very few students, including Sarah, do at the outset. Initially, calculations tend to be all over the page, making it hard to recreate the logic. Only as students begin to focus on the steps they take, not the numbers they produce, do they begin (often enough spontaneously) to record those steps in a re-readable way.

Paradoxically, urging students to write neatly and show all their work can backfire. "Write neatly" gets some students to erase their work or



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THE AIM IS TO COME TO UNDERSTAND THE FLOW OF THE CALCULATIONS—THEIR NATURE AND ORGANIZATION, THEIR REGULARITY—WELL ENOUGH TO INVENT A GENERAL FORMULA OR SOLUTION METHOD.

hide it on scrap paper. "Show your work" gets some students to camouflage their reasoning with extra steps and calculations that they did not need. Mostly, it takes time and experience, but you can help. Your genuine curiosity about how students got their results can encourage them to explain, recreating the steps on the spot and perhaps noticing how scattered and unclear their records were. If you take notes as they explain, your notes can clarify their logic and model a new style for them.

Sarah's class is used to this way of thinking, so the process does not take Sarah long. After a third try, she sees the common structure even before finishing the arithmetic. Each time, she picks a

Guess: 40 x new
$- 20 \times \cancel{\text{old}}(\text{new} - 3.50)$
?? 370
oh! :) $40n - 20(n - 3.50) \stackrel{?}{=} 370$

Fig. 4 Sarah generalizes and writes an equation containing the new salary, *n*.

new salary, computes the new earnings (40 × new), computes the old salary (new – 3.50), computes the old earnings (20 × (new – 3.50)), and subtracts to see if she gets 370. See **figure 4**.

Using n as an abbreviation for "new," she writes

$$40n - 20(n - 3.50) \stackrel{?}{=} 370.$$

The $\stackrel{?}{=}$ symbol preserves the idea that this equation arose as a test of equality. In some mathematical statements, like $f(x) = x^2 - 7$, "=" is an assertion, not a question. The "$\stackrel{?}{=}$" notation in

$$40n - 20(n - 3.50) \stackrel{?}{=} 370$$

is a reminder that, in this mathematical statement, the symbol means "under what conditions is this statement true?" The conditions for truth are called solutions. In our classes, we sometimes use "$\stackrel{?}{=}$" to emphasize the question. Sarah is checking to see when the two sides are equal.

In this problem, Sarah could bypass algebra to find a numerical solution directly, using guess-and-check, but that is neither her goal nor the goal for her learning. She is asked to find an equation, not just a numerical answer, and she is learning to use the guess-check-generalize approach precisely to learn how to get that equation. Much of mathematics, in fact, is only about the equation, and not about numbers.

Creating a Formula

Give the equation for a circle with center at (3, −2) and radius 7.

This problem is often posed as an exercise in applying an already-known formula, not as a problem to solve, but the students in this class were being asked to figure out an equation for a circle without having first been taught one.

Ben uses the same experiment-driven approach that Sarah used. He and his classmates have not yet learned equations for circles, but they do know that all points on a circle must be the same distance

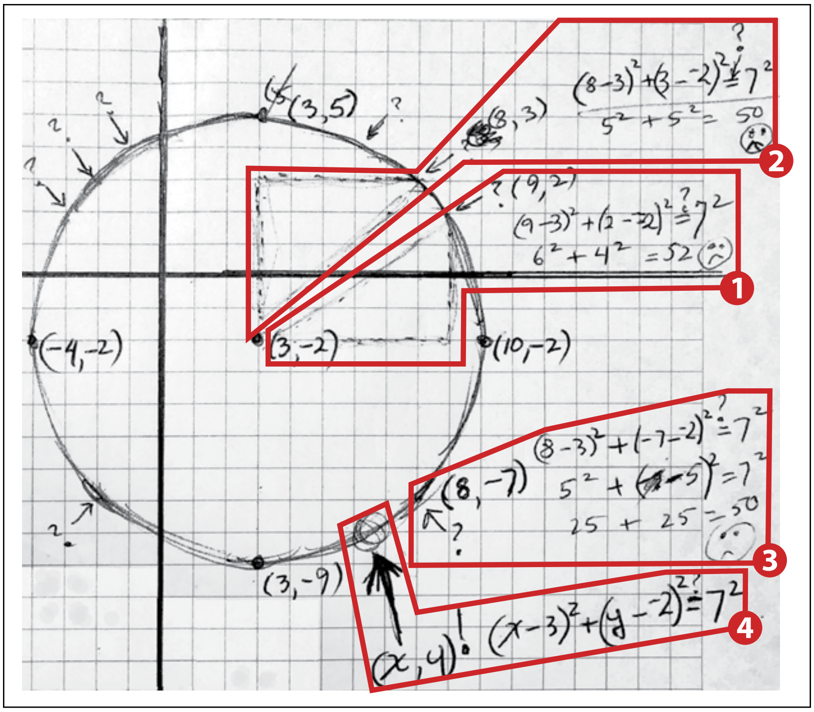


Fig. 5 Ben sketches and calculates.

from the center—in this case, 7 units—and they know how to use the Pythagorean theorem to compute the distance between points.

Ben first makes a sketch, starting with the four points 7 units above, below, left, and right of the center. He roughly measures several other points to help make a good circle. He did not have to be so neat, but he said that he thought it might give him some insights.

It didn't.

So Ben starts testing points (see **fig. 5**). As the whole class has been learning, any points will do, just as any guessed hourly wage would do in the Tamara problem. It doesn't matter whether the points he guesses turn out to be on the circle or not; the goal is to determine the equation. For ease of calculation, he chooses only integer coordinates. This is natural—students tend to choose integers unless there is reason not to—but the strategy for choosing numbers evolves in class: Choose easy numbers to gather data to build a theory; go "extreme" to test a theory. Over the course of the year, Ben has learned to record his calculations in a way that reveals the logic behind them. Like Sarah, his calculations are less scattered than they used to be.

Ben chooses plausible candidates, even though that isn't necessary. The first candidate he chooses is (9, 2). See region 1 in **figure 5**. Scaffolding his use of the Pythagorean theorem, he sketches the right triangle he will use to test the distance from his guess-point to the circle's center. He computes the lengths of the triangle's legs by subtracting coordinates: $(9 - 3)$ and $(2 - {}^{-}2)$; squares those lengths; adds them, writing $(9 - 3)^2 + (2 - {}^{-}2)^2$; and checks to see if that is 7^2.

Our wording of Ben's method could be translated directly into an equation. In fact, Ben's own writing makes it look as if he already saw the general equation form, but apparently he had not yet fully recognized that regularity.

Writing $(9 - 3)^2 + (2 - {}^{-}2)^2$ is an enormous step that Ben and his classmates have had to learn. Beginners at using this guess-check-generalize method tend to have discrete computations—scrawls like $9 - 3 = 6$ and $2 - {}^{-}2 = 4$, and $6^2 = 36$—spread all over separate scratch paper or even erased, and tend to record new steps, like $36 + 16$, without preserving the structure $(9 - 3)^2 + (2 - {}^{-}2)^2$ that led to them. But Ben and his classmates are now fairly used to this way of thinking.

So Ben performs the calculation with a new point, (8, 3). He ends up with the "point-tester" equation $(8 - 3)^2 + (3 - {}^{-}2)^2 \overset{?}{=} 7^2$. See region 2 in **figure 5**. The fact that he then actually performs the computation rather than just recording it seems to show that he has not yet seen the generality in this equation: He is still testing to see if this is, by lucky chance, an equality (meaning that the point is on the circle).

Ben tests yet a third time, this time with point (8, –7), and writes $(8 - 3)^2 + ({}^{-}7 - {}^{-}2)^2 \overset{?}{=} 7^2$ (see **fig. 5**, region 3).

Then he stops. The three equations he has written have a common structure. He sees which parts vary and which remain the same, and so (in **fig. 5**, region 4) he tests the "generic point" (x, y) the same way. As always, this generic formulation, without the question mark, is the goal equation. It never mattered what points Ben chose for his experimentation; the point (100, 200) would have worked just as well. Checking any points makes Ben's reasoning visible so that he can generalize how that checking process works.

The repetition is not done to practice a formula—Ben did not even know a formula. Nor is it done to practice calculating. The repetition is a set of concrete experiments to reveal the process that underlies them and to build a general solution. When students understand—by generating—each element of their equation, it's a small step for them to change $(x - 3)^2 + (y - {}^{-}2)^2 = 7^2$ to the generic solution $(x - C_x)^2 + (y - C_y)^2 = r^2$.

THE REAL VALUE OF MP 8

Here is the real value of MP 8: The guess-check-generalize method is general. It works for word problems; it works for finding the equation of a circle or a line through two given points; it works for finding a formula that computes the area of a triangle given only its three side-lengths; and it works for solving mathematical problems beyond high school. The general method is to "look for and express the regularity in repeated reasoning." The experiment consists of choosing a (plausible or implausible) candidate solution and reasoning out how to check to see if it is a solution. Performing several such experiments helps one distinguish the moving parts from the parts that stay the same. Replacing the moving parts with variables then makes the solution-checker generic.

DISTINGUISHING MP 7 AND MP 8

People are sometimes unsure what distinguishes MP 8 (look for and express regularity in repeated reasoning) from the often closely intertwined aspect of mathematical practice MP 7 (look for and make use of structure). Expressing the regularity in repeated reasoning (MP 8) depends on recognizing some structure (MP 7), and though these two often work in tandem—as they did in both Sarah's and Ben's work—there is an important distinction.

MP 7 focuses on seeking and using structure in a mathematical object, such as an expression, function, equation, geometric construction, or sequence. MP 8 focuses on the reasoning—a process, a sequence of steps, not an object.

Here is an example. We can, of course, solve an equation like

$$14(x - 2) - 5 = 13(x - 2)$$

by multiplying through, gathering like terms, and so on, but if we read its structure as

$$14\blacksquare - 5 = 13\blacksquare$$

we see that $\blacksquare$ must be 5. That conclusion translates back to $(x - 2) = 5$, so x must be 7. Looking for structure can give insight into a problem, and using the structure can make the problem easier to solve. In some cases, structure is a necessary clue. If we see $x^6 - y^4$ as a difference of squares, $\heartsuit^2 - \bigstar^2$, with $\heartsuit = x^3$ and $\bigstar = y^2$, then we can factor it as $(x^3 + y^2) \bullet (x^3 - y^2)$.

Ben and Sarah used structure. They could see the regularity in their reasoning because their records preserved the structure, and not just the result, of their calculations. No pattern would be apparent if they had recorded only the resulting numbers. Sarah's parenthetical note, shown in the second line of **figure 2**, "\$6.50 = old (new – 3.50)," records the logic that produced \$6.50.

Ben's use of $(9 - 3)^2 + (2 - {}^-2)^2$ and not just $36 + 16$ records the structure, preserving his line of reasoning. Also, Ben's sketched-in auxiliary lines revealed right triangles for which the Pythagorean theorem was useful. This use of structure (MP 7) supported Ben's repeated reasoning (MP 8) about which points are on the circle and which are not.

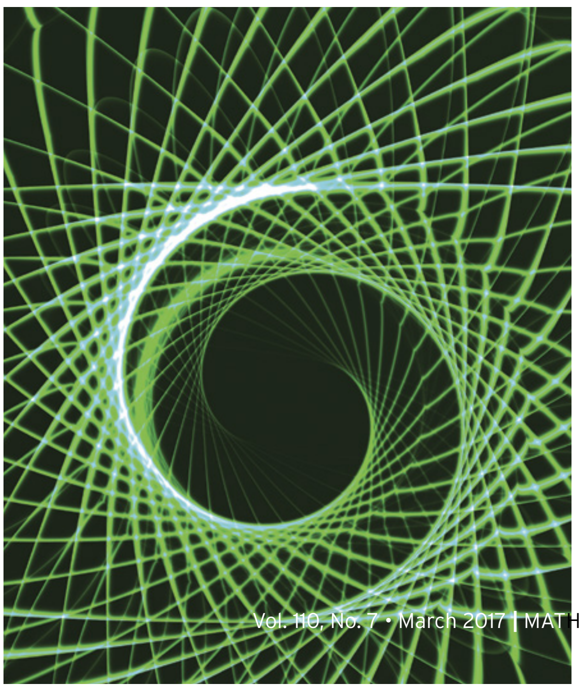
But these two mathematical habits of mind are independent. Analyzing $14(x - 2) - 5 = 13(x - 2)$ as $14\blacksquare - 5 = 13\blacksquare$ requires no repeated reasoning. The same is true when Ben imposed the structure of a right triangle on the distance between two points. Each is a case of finding (or imposing) the structure in a mathematical object, a single case.

MP 8 involves abstracting a generic argument from a few examples of the reasoning applied concretely. As in Ben's and Sarah's work, regularity in repeated reasoning (MP 8) is a process that we use to *generate* a mathematical object, an equation, not (as in MP 7) an *analysis and use* of the object.

ELEVATING THE HABIT BEYOND ALGEBRA

Students are often taught methods for translating word problems into symbols without using experimentation and repeated reasoning. For many students, those methods do not come easily; and for many problems, like Ben's, they do not work at all. When facing hard problems, mathematicians naturally turn to experiments, looking for regularities that can lead to a theory. This process is not

LOOKING FOR REGULARITIES THAT CAN LEAD TO A THEORY IS NOT NATURAL TO BEGINNERS— IT MUST BE LEARNED.



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natural to beginners—it must be learned—but becoming proficient in mathematics requires this way of thinking. We make it a habit; it becomes second nature—a mathematical habit of mind. That is why CCSS elevates the process to a standard.

MP 8, like all the mathematical practice standards, applies across mathematical domains and in all grades, but the CCSS writers explained MP 8 largely in the context of algebra. Perhaps that is because a key part involves expressing the regularity of repeated reasoning and algebra is such a powerful tool for expressing general cases. A full discussion of MP 8 outside of algebra is beyond the scope of this article, but you can find examples and discussion of MP 8 in other contexts in http://mathpractices.edc.org/. This habits-of-mind approach to mathematics is at the core of certain curricula (e.g., EDC 2013; Mark et al. 2014) and described in depth in work by Goldenberg and colleagues (2015).

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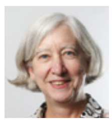

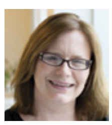

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