Research on Technology in Mathematics Education: Theoretical Frameworks and Practical Examples

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Introduction

We now live in the 21st century, although you might not realize that fact if you were a student sitting in a "typical" mathematics class in most rural or urban school districts in the USA. Outside of school, technology tools and their applications are an integral part of modern life. We use and depend on them for entertainment, information, communication, transportation, commerce, research, comfort, shelter, safety, food production, medical treatment, as well as creative, self-expression and social networking.

At the beginning of the 21st Century in the USA, the National Council of Teachers of Mathematics (NCTM) made a very strong recommendation for the integration of technology in the teaching and learning of mathematics by including the *Technology Principle* as one of the six principles that frame the NCTM *Principles & Standards for School Mathematics* (NCTM, 2000). The Technology Principle states: "Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning." This strong claim has research evidence to support it but more research is needed to understand the affordances and constraints that affect this principle's implementation, both inside and outside of formal school settings.

The available technologies for teaching and learning, both in and out of school have expanded tremendously during the first decade of this century. Alongside computers and calculators we have iPods, iPhones, and now iPads; hand-held computing devices such as the TInSpire (which operates more like a computer than a calculator); networked calculators; wireless response systems; scientific probes that can be connected to hand-held devices or computers for generating real data in real time (CBL's and CBR's); Interactive SMART Boards and now, interactive SMART Tables for collaborative problem solving activities. The explosion in web-based resources for finding information, for social networking, for entertainment and for collaborative problem solving in on-line communities has changed the way we live our lives – outside of school. Perhaps one powerful reason for why almost a third of the students entering high schools in this country "drop out" before completing their high school diploma (Gonzalez, 2010) is that education in many schools is presented in the same way as it was in the 19th and 20th centuries. The educational process in school bares little resemblance to how people learn outside of school.

As educators, we need to investigate how children and young adults are making use of the technological environment in which they live and what they are learning from that use. As mathematics educators, we need to understand how we might harness this technological environment to enhance the learning and teaching of mathematics – both in-school and out-of-school.

Needed Research on Technology in Mathematics Education

Research on Technology in Mathematics Education needs to encompass many dimensions and address important questions related to human development in our technological world. The following list is one possible starting point. I shall elaborate on each of these points throughout the paper and demonstrate examples during my presentation:

- *Learning:* How and what do students learn through use of technology?
- *Teaching:* How and what do teachers teach using technology?
- *Curriculum:* What mathematics can and should be accessible through the use of technology?
- *Design of Technology:* How does the specific interface design of a technology impact its use?
- *Use of technology:* Actual use may differ from the designed use – how do the different uses affect learning and teaching outcomes?

- *New Media for learning:* New networking and social interaction technologies offer new media for learning both inside and outside the classroom. How and what kind of learning may take place in these new media?
- *New Media for teaching:* New networking and social interaction technologies offer new media for teaching both inside and outside the classroom. How and what kind of teaching may take place in these new media?

Theoretical Frameworks for Addressing Research on Technology

What are appropriate theoretical frameworks for investigating all of the above? This question was the focus of several chapters in the recently published 17th ICMI Study: *Mathematics Education and Technology—Rethinking the Terrain* edited by Celia Hoyles and Jean-Baptiste Lagrange (2009). In Chapter 7, Paul Drijvers, Carolyn Kieran and Maria-Alessandra Mariotti (with 8 members of their working group) provide an historical overview of theoretical perspectives they consider relevant to integrating technology into mathematics education (Drijvers, Kieran & Mariotti, 2009). They pay particular attention to *Instrumentation Theory* and *Semiotic Mediation,* but make "a plea for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives." (p. 89)

Instrumentation Theory

Many European researchers (especially in France) have adopted and adapted *Instrumentation Theory* (Verillon & Rabardel, 1995) for their research on the use of technological tools. Central to this theory is the process of *Instrumental Genesis* -- How a tool changes from an artifact to an instrument in the hands of a user, and how both the tool and user are transformed in the process. Kathy Heid (2005) describes *instrumental genesis* as the development of a working relationship between the user and the tool:

> One needs to be careful not to give the impression that technology itself makes the difference in teaching and learning. It is, of course, not the technology that makes the difference but rather how it is used and by whom. Those who have studied the use of technology in mathematics teaching and learning have noted that technology mediates learning. That is, learning is different in the presence of technology. The representations that students access may conceal or reveal different features of the mathematics, and the procedures students assign to the technology (as opposed to doing them by hand) may affect what students process and learn. Moreover, how a student uses technology is dependent on his or her ever-

> changing relationship to the technology. When a user first encounters a particular technological tool, his or her uses of the technology may be confined to rote application of the specific keystrokes or procedures that had been introduced. As the student develops facility with, and an understanding of, the capabilities of the technology, the technology becomes an instrument that the student can tailor flexibly to specific needs. (p. 348)

Kieran and Guzmán (2005) describe this process as follows: "A tool, which starts out merely as an artifact, becomes an instrument for the user only when he or she has been able to appropriate it for himself or herself and has integrated it fully within his or her activity." (p. 36) They explain further: "in this process of transforming the artifact into an instrument, the learner is not just simply learning tool-techniques that permit him or her to respond to given mathematical tasks. Mathematical concepts codevelop [sic] while the learner is perfecting his techniques with the tool." (p. 36). Following from the work of Artigue (2002) and Lagrange (2000) they adopt a dialectical interaction triad of *Task*, *Technique*, and *Theory* that serves as their conceptual framework for their study of the role of technology in the development of mathematical thinking. They propose that tool-techniques constitute a bridge between tasks and the emergence of theoretical (i.e. mathematical) knowledge. "It is by looking at the *techniques* that students develop with their technological instruments, in response to certain tasks, that we obtain a window into the evolution of their mathematical thinking." (Kieran & Guzmán, 2005, p.36)

Semiotic Mediation

The notion of *semiotic mediation*, according to which cognitive functioning is intimately linked to the use of signs and tools, and affected by it was introduced by Vygotsky (1978). Elaboration of this notion with respect to both mathematics learning and the use of technological tools has proved to be a useful theoretical framework for many researchers (Saenz-Ludlow & Presmeg, 2006). Drijvers et al. (2009) describe a semiotic approach to mediation as follows:

> The mediating potential of any artifact resides in the double semiotic link that such an artifact has with both the meanings emerging from its use for accomplishing a task, and the mathematical meanings evoked by that use, as recognized by an expert in mathematics. In this respect, any artifact may be considered both from the individual point of view – for instance, the pupil coping with a task and acting with a tool to accomplish it – and from the social point of view – for instance, the corpus of shared meanings recognizable by the community of experts, mathematicians or mathematics

> teachers. From a socio-cultural perspective, the tension between these two points of view is the motor of the teaching-learning process centered in the use of an artifact. (pp. 116-117)

Thus any artifact (including those belonging to our new technologies) may offer valuable support to the learning of mathematics according to its *semiotic potential*. How we identify that potential might require different approaches (Bartolini Bussi & Mariotti 2008). Drijvers et al. (2009) suggest that an "a priori analysis [of the double semiotic relationship], involving in parallel two interlaced perspectives, the cognitive and the epistemological" (p. 117) may lead to the identification of the semiotic potential of an artifact that can be related to particular educational goals. These researchers recommend that the determination of the semiotic potential of any learning tool should be an element in the design of any pedagogical plan centered on the use of that tool. They also suggest that:

> The construct of instrumental genesis, discussed above, provides a crucial contribution to such analysis. As long as the evolution of personal meanings is related to the accomplishment of a task, it can be analyzed in terms of instrumental genesis, that is, meanings may be related to specific utilization schemes that themselves are related to the specificity of the tasks proposed to students. Thus, an instrumental approach becomes fundamental not only in the identification of semiotic potential, but also in the design of appropriate tasks, as well as in the interpretation of pupils' actions and 'speech' acts. (p. 117)

Integrative Frameworks

The frameworks discussed above focus mainly on the interaction of the learner and the tool. We need to take into account the role of the teacher (or more experienced other) in the didactical situations made possible by the integration of technology. In Chapter 8 of the ICMI study, Olive and Makar (2009) along with 4 members of their working group, focus on the mathematical knowledge and practices that may result from access to digital technologies. They put forward a new tetrahedral model derived from Steinbring's (2005) didactic triangle (see Figure 1) that integrates aspects of instrumentation theory and the notion of semiotic mediation. This new model illustrates how interactions among the didactical variables: student, teacher, task and technology (that form the vertices of the tetrahedron) create a space within which new mathematical knowledge and practices may emerge. Olive and Makar state "It is not arbitrary that we place the student at the top of this tetrahedron as, from a constructivist point of view, the student is the one who has to construct the new knowledge and develop the new practices, supported by teacher, task and technology." (p. 168)

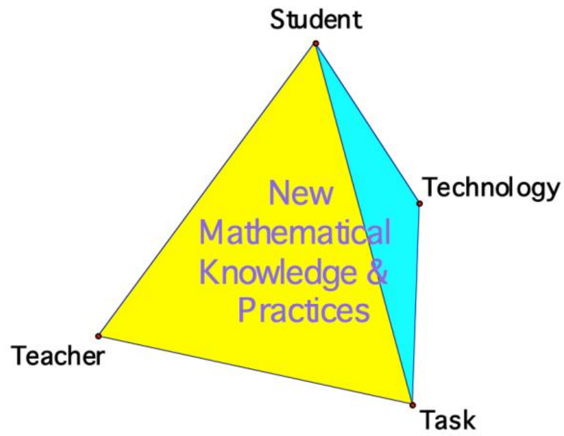

Figure 1: The Didactical Tetrahedron (from Olive & Makar, 2009, p. 169)

In her dissertation study of three high school mathematics teachers teaching with technology, Hyeonmi Lee (2010) focused on the teacher vertex of this tetrahedron. She incorporated Goos' (2005) modification of Vygotsky's construct of the Zone of Proximal Development (ZPD) as it relates to teachers' beliefs, knowledge, and skills in working with technology. She also included Valsiner's (1987) additions to Vygotsky's ZPD, the Zone of Promoted Action (ZPA) and the Zone of Free Movement (ZFM) in her theoretical model for teaching with technology. Based on Goos and Soury-Lavergne's (2009) application of these zones to teachers teaching with technology, Lee investigated the participant teachers' teaching from these three perspectives:

- ZPD: teachers' beliefs, knowledge, and skills in working with technology
- ZPA: teacher education, professional development, and teaching experience with colleagues
- ZFM: access to hardware, software and laboratories, access to teaching materials, support from colleagues, curriculum and assessment requirements, and students' attitudes and abilities. (p. 22)

Figure 2 illustrates Lee's view of how these three zones impact on the teacher's use of technology.

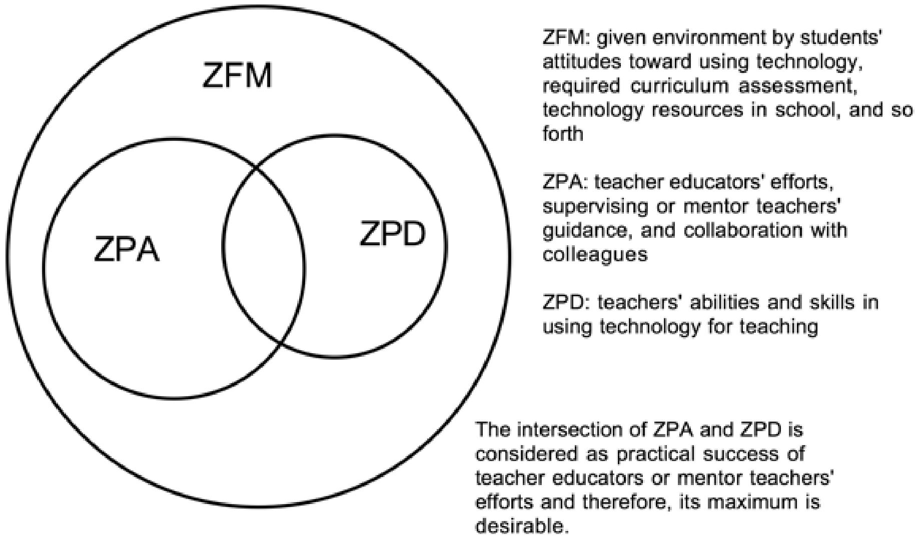

Figure 2: Three metaphors of ZPD, ZPA, and ZFM in working with technology (from Lee, 2010)

Lee went further with her theoretical model, incorporating modifications of Zbiek et al.'s (2007) didactic pyramid for teachers as learners in technology-integration courses. She further investigated how this pyramid influences the three zone theories that she placed as vertices of a tetrahedron, with the teacher as the fourth vertex and teaching emerging from the interactions among all four vertices (a modification of the Olive & Makar tetrahedron shown in Figure 1 above). Figure 3 illustrates Lee's model for the focus of her dissertation study.

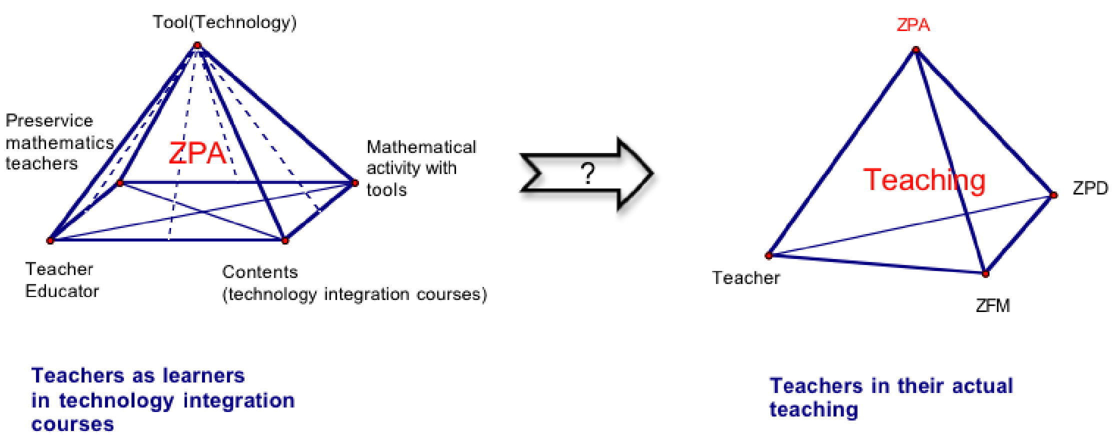

Figure 3: Lee's model for the focus of her study (from p. 23)

The focus of her study is represented in Figure 3 by the double arrowed box containing the question mark. Thus, the study focused on the ways in which the experiences represented by the pyramid on the left related to the observed and perceived aspects of the teaching situation represented by the tetrahedron on the right. I consider Lee's model as one possible answer to the plea that Drijvers et al (2009) made for the development of integrative theoretical frameworks that allow for the articulation of different theoretical perspectives.

Complexity Theory

Davis and Simmt (2003) applied a theoretical framework from complexity science to the teaching and learning of mathematics, promoting the shift away from mathematics as content towards the emergence of a mathematical community as a learning system. They proposed five conditions as necessary for this emergence of a learning system:

1. Internal diversity: but not the kind of diversity achieved by structured group work or other formal classroom organization strategies, because "diversity cannot be assigned or legislated, it must be assumed – and it must be flexible" (Davis & Simmt, 2003, p. 149)
2. Redundancy: which provides the necessary degree of sameness to allow people to interact while compensating for each other's weaknesses
3. Distributed control: acknowledging that the locus of learning is in the collective rather than the individual
4. Organized randomness: establishing the enabling constraints necessary for generative activity
5. Neighbour interactions: providing sufficient density of interactions between agents to open up new conceptual possibilities (from Goos & Soury-Lavergne, 2009, p. 320)

Margaret Sinclair examined three different mathematical activities that incorporated the use of technology with respect to the five conditions above (Goos & Soury-Lavergne, 2009). The only one of the three tasks that met all the conditions for the emergence of a learning community was an independent study by her students that made use of a variety of technological applications. Davis and Simmt (2003) argue that "emergent events cannot be caused, but they might be occasioned" (p. 147). Goos and Soury-Lavergne (2009) point out that:

> The difference here is between tasks that are *prescriptive* (specifying what is *permitted*; everything else is forbidden) versus *proscriptive* (specifying what is *forbidden*; everything else is allowed); in other words, emergence requires enabling constraints. (p. 320)

Sinclair's successful task was a proscriptive rather than prescriptive task.

While Goos and Soury-Lavergne were looking at *Complexity Theory* as a suitable theory for examining the teacher's role in technology integration in the classroom, I see it as being more broadly applicable to learning situations outside the classroom, especially to on-line communities of practice and web-based multi-player gaming scenarios.

Research Methodologies for Studying Technology Integration

Appropriate methodology for any research study depends primarily on the research question to be investigated and the theoretical framework within which the research is being conducted. I look with great skepticism on studies that purport to test the "effect of technology on student learning" by trying to set up experimental-control designs that are supposed to "isolate" technology as a measurable variable in the teaching-learning situation. As stated above by Kathy Heid (2005) "One needs to be careful not to give the impression that technology itself makes the difference in teaching and learning. It is, of course, not the technology that makes the difference, but rather how it is used and by whom." (p. 348)

The list of questions I posed in my introduction suggest a variety of methodologies that might be used in attempting to address these questions.

Learning: From a constructivist perspective, research related to how and what students learn through use of technology should aim to build models of "epistemic students" (Steffe, 2010a) that would be useful for teachers, parents, software designers, designers of learning environments and policy makers. As Steffe points out in his plenary paper for the WISDOM^e conference, the researcher needs to be engaged with the students in a teaching-learning situation in order to build second-order models of students' ways and means of operating when engaged in challenging mathematical tasks with the aid of technology.

Teaching: Research investigating how and what teachers teach using technology has mainly focused on the teacher in the classroom, using observational methodologies derived from ethnography. Goos and Soury-Lavergne (2009) make the argument that:

> (1) teacher characteristics (their mathematical and pedagogical knowledge, beliefs and attitudes, skill and comfort in using digital technologies), (2) institutional contexts (access to resources, policy pressures, curriculum change), and (3) professional learning and development influence the integration of digital technologies into mathematics teaching. (p. 327)

This broad view of the influences on teaching with technology requires methodologies that go beyond observation. Institutional analysis, belief questionnaires, in-depth interviews, reflective journaling, and video portfolios can all contribute to the study of teaching with technology.

Curriculum: Determining what mathematics can and should be accessible through the use of technology (and thus included in curriculum recommendations) should be a by-product of the studies on learning and teaching. Historically, this has not been the case. Rather, curriculum development and research has proceeded from an analysis of the structure of mathematics as perceived by adult mathematicians (see Steffe, 2010a for the deficiencies and dangers of this approach).

Design of Technology: Research on how the specific interface design of a technology impacts its use is critical for understanding the ways in which humans interact with the technology. Nicholas Jackiw, the designer of the *Geometer's Sketchpad* makes the following point:

> [D]esign certainly acts as the first doorway and first doorkeeper to any deeper curricular or epistemological innovation an educational technology might offer. For it is not at the structural level, but rather on the surface—at the designed *interface*—that users interact with technologies; that meanings are negotiated; that cognitive, psychological, educational, and social transformation may, or may not, occur. (Butler, Jackiw, Laborde, Lagrange & Yerushalmy, 2009, p. 432)

Jackiw suggests methodologies from the recent field of human computer interaction, and within this field, the study of interaction design, focuses precisely on the ways "in which software signifiers are consumed by users, and on how users' conceptual models of technology artifacts grow and change in response to interaction." (Butler et al. p. 433)

Use of technology: Investigating how the different uses of technology affect learning and teaching outcomes should also be a focus of the research on learning and teaching with technology, as well as a concern of the design research. Methodologies that focus on *how* the technology is used, by students and teachers, and not just whether it is used or not, are essential for understanding any affects on outcomes. Instrumentation theory provides an appropriate theoretical framework for investigating how technology is used and shaped by that use.

New Media for learning: New networking and social interaction technologies offer new media for learning both inside and outside the classroom. Hoyles et al. (2009) pose the following questions regarding these new interactive media:

- What is the potential for creating virtual communities for mathematics learning and permitting communication between individuals from different educational settings?
- What is the potential contribution to mathematics learning of different levels of interactivity and different modalities of interaction, and how might this potential be realized?
- What is special about the potential of collaborative study of mathematics whilst physically separated, and how might this potential be harnessed so as to support mathematics learning? (p.440)

The media themselves offer new methodologies for investigating such questions. Because these interactions take place in a digital medium, they can be easily recorded or catalogued for many purposes. Indeed, this is already happening with our use of credit and debit cards and the electronic scanning of all of our purchases. Every single item we purchase from the grocery store adds to our profile in the corporate database, informing the grocery chain of our specific preferences and using this information to focus promotional coupons and email messages specifically for us. Students' interactions in a digital learning (or gaming) medium could also be recorded and cataloged in ways that could provide the researcher with data for rich analyses of learning trajectories, modes of communication and collaboration, the emergence of different problem solving strategies, as well as patterns of actual use. *Semiotic mediation* and *Complexity Theory* could provide appropriate frameworks for such analyses. Also Shaffer's (2006) theory of *epistemic frames* that characterize the situated understandings, effective social practices, powerful identities, shared values, and ways of thinking of important communities of practice can be a useful framework for analyzing these digital records. In his paper on video games and the future of learning, Shaffer et al. (2008) make the following point:

> To build such games requires understanding how practitioners develop their ways of thinking and acting. Such understanding is uncovered through *epistemographies* of practice: detailed ethnographic studies of how the epistemic frame of a community of practice is developed by new members. That is more work than is currently invested in most "educational" video games. But the payoff is that such work can become the basis for an alternative educational model. (p. 12)

New Media for teaching: New networking and social interaction technologies offer new media for teaching both inside and outside the classroom. Research on how and what kind of teaching may take place in these new media has focused primarily on distance learning techniques. When distance learning was first introduced, it mimicked the face-to-face lecture modality and was seen as a poor substitute for the real thing. Distance learning is rapidly evolving into a dynamic medium for engaging students at a distance, in both synchronous and asynchronous modes. Social networking platforms, such as “Second Life” are being used by universities to create virtual learning communities in which students and teachers interact via on-screen avatars. Undertaking Shaffer’s epistemographies of practice in such virtual teaching and learning communities could help us understand how teaching is transformed in these new media.

Example Implementations of Technology in Mathematics Education

In this section of the paper I will attempt to briefly describe some examples of technology integration in mathematics education that address several of the questions I posed in the introduction.

The Dynamic Number Project

This project will eventually address almost all of the questions I have posed in this paper. Key Technologies (developers of Dynamic Geometry and Dynamic Statistics software) have been awarded a research and development grant from the National Science Foundation to develop *Dynamic Number* (DN) tools for students and teachers in elementary and middle school. In the NSF proposal for the project, Scher and Rasmussen (2009) make the following points:

> Currently, Dynamic Number ideas only exist in highly controlled, narrowly content-focused “applet”- like incarnations. Alas, these interactive models are useful only at the rarest triple concurrence of technological availability, curricular relevance, and student need. Furthermore, they are capable of being built only by those equally rare individuals who combine curriculum development expertise with sufficient technological prowess and appropriate professional contexts to pursue such work. If, instead, these ideas became infrastructure in a general-purpose mathematical tool accessible not only to curriculum developers but to students and teachers, **the educational, technological, and social school conditions are ripe for Dynamic Number technology to have broad and transformative impact, at a national scale, on students’ mathematical understanding and performance relating to core number constructs, elementary number theory, and early algebra ideas across the grades 2–8 curriculum.** (p. 2 emphasis in the original)

Actualizing this potential is the goal of the Dynamic Number Project. During the first year of the project, 19 teachers in grades 2 through 8 were recruited across the USA and were provided with six weeks of on-line training in the use of *The Geometer's Sketchpad* (GSP 5) and initial prototypes of Dynamic Number tools constructed in GSP. In the summer of 2010, four more teachers in Croatia joined the project. My role in the project is as director of the evaluation component. Along with Les Steffe and a doctoral student, Doug Griffin at the University of Georgia, we have participated in the on-line community established for on-going communication among the participating teachers and developers, reviewed the initial prototype tools and provided feedback to the developers on possible ways to improve or adapt these prototypes, and conducted interviews with the five teachers in Georgia. In the interview that Doug Griffin conducted with one of the Georgia teachers, following the six-week training, the teacher reported that she had already used a couple of the dynamic number activities with her fourth grade students. The following is an excerpt from the verbatim transcript of that interview:

> I've done two, I used the same program. I did the dividing and subdividing fractions on a number line sketch with my children. Just to introduce them to what is ½? What is 1/3? I found it VERY interesting because I felt like I taught several things in conjunction and didn't just address what does a fraction look like. They were able to…my intent was to teach equivalent fractions. But even more than that, one of the most beneficial things I thought from the lesson was, *Sketchpad* only has up to 1/6, so we had to make 1/10 and my kids had to develop that mathematical thinking: How can I make 1/10 when I only have ½, 1/3, ¼, 1/5, and 1/6? And 1/9 was the most interesting to create, because all of my kids wanted to start with ½. They told me "You need to split the number line in half" and then they would say "Ok, now use fourths up to the half mark and then use fifths up until the end". Then they would say "No, that doesn't work because fractions have to be equal". So then I said "OK let's start over"! Then they said "Let's use the ½, the 1/3, and the 1/6" (laughter). I was able to actually show that to them, it was a visual representation of what they were saying. And that taught fractions how to be equal better than anything I think, because then they would say "Let's just use 1/6 and 1/3 instead of splitting it in half". So I would drag 1/6 about ¾ of the way, then I would take 1/3 and take it about halfway, not even complete the number line, and they would say "No you can't do that either because they are still not equal". Finally they developed that it was 1/3 and 1/3 and 1/3, but it took a while, and I think that was a very………..insightful moment for them.

[In the presentation, I will demonstrate the Dividing and Sub-dividing activity that the teacher described in this transcript as well as other DN models.] The teacher's enthusiasm for use of this DN tool, if only in a demonstration mode, and the skillful way in which she used it to provoke perturbations in the students' reasoning about fractions, are early indications of possible changes in teaching and learning that we are currently analyzing and documenting in this project.

During the second year of the project we have been videotaping the teachers' class lessons in which they make use of the Dynamic Number (DN) tools and also conducting interviews with individuals and small groups of students. In these interviews the researchers used DN tools with students to explore students' conceptions of units, measurement, fractions and variables. Preliminary results have been presented at the annual conference of NCTM in the USA and at the *35th Annual Conference of the International Group for the Psychology of Mathematics Education* (PME 35) in Ankara, Turkey (Olive, 2011).

This year, the Dynamic Number project is testing interactive GSP sketches for children to explore number concepts at home or in school, using the new *Sketchpad Explorer* that was recently developed by Key Technologies for exploring GSP sketches on the iPad. Initial use of the first interactive sketch with children in third and fourth grades indicates that it is a powerful motivator for engendering children's construction or recall of multiplication facts. In my presentation I will share a short video of two fourth graders playing with this first sketch (*Hop Along*) on an iPad.

The DN Project is based on well-established theoretical frameworks, integrating the Elkonin-Davydov (Elkonin and Davydov, 1966; Davydov et al., 1999) approach to elementary mathematics with Steffe's notion of a "connected number" (a connected but segmented linear unit). Steffe's connected number constitutes a step in unifying discrete and continuous quantitative schemes because it opens the possibility that a connected but segmented unit can be a situation of the child's counting scheme. Steffe's (2010b) description of the construction of a "Connected Number Sequence" provides strong support for the Dynamic Number approach:

> The construction of a connected number sequence is an initial step in the construction of measurement as well as an important step that integrates discrete and continuous quantity. A child at this level has constructed an awareness of indefinite length as well as of indefinite numerosity as quantitative properties of a connected number. Hence, I regard both an awareness of length and an awareness of numerosity as extensive quantities, which generalizes the concept not only across the discrete and the continuous, but also across the schemes that constitute measurement and number. (p. 56)

The evaluation component of the project will feed into the design-research component on a cyclical basis, thus the design of the DN tools will be shaped by our work with teachers and students as they make use of the prototypes. We anticipate uncovering several cycles of instrumental genesis through this process.

Dynamic Geometry Environments

In a dynamic geometry environment (DGE), geometric objects are constrained by their geometric properties (unlike paper-and-pencil sketches that can be distorted to fit expectations) similar to how physical objects are constrained by properties of physics when manipulated within the world. By observing properties of invariance simultaneously with manipulation of the object, there is potential to bridge the gap between experimental and theoretical mathematics as well as the transition from conjecturing to formalizing. As Laborde et al. (2006) state, DGE "has provided access to mathematical ideas by allowing the bypassing of formal representation and access to dynamic graphing which is particularly important for the learning of geometry. … Just as digital technology provides means to by-pass formalism, it may also provide the means to transform the way formalism is put to use by students." (p. 284)

Dynamic geometry environments have been available for more than 20 years, and much has been written about them; many research efforts have investigated their use with students in classrooms from kindergarten to college mathematics and mathematics education courses. The 17th ICMI Study volume (Hoyles & Lagrange, 2009) has no less than 50 listings for DGE in its index (p. 488) and three of those 50 are page-ranges that bring the total to 68 pages in which DGEs are discussed. I shall not attempt to review this massive corpus of data in this paper but refer the reader to this published volume for references to important research on DGEs.

The continued development of DGEs and the relative success in their widespread use in classrooms around the world provide lessons for future design of educational technology. Jackiw (2009) makes the following comment regarding the importance of design in the eventual influence of technology on mathematics education:

> To the degree our work in mathematics technology aspires to educational influence at significant scale, rather than just to the pleasure of small, pre-qualified technological elites, we have first to admit that design matters—that specific design matters, specifically—and, second, to develop a much richer discourse for design analysis. (p. 432)

One of the most challenging design problems with respect to dynamic geometry is that of interacting on a 2-D surface to manipulate representations of 3-D shapes. Currently, only one of the major DGE groups has addressed this challenge with the development of Cabri 3D. Jean-

Marie Laborde (2009) indicates that the challenge is to find the right metaphors to help people reinvest their existing body of knowledge in order that they feel familiar with the new environment:

> In *Cabri 3D* we decided to use conic perspective as [the] default perspective. Precisely, objects are represented as they would be seen in the hands of the user at a distance of 40 or 50 cm from his/her eyes. We call this perspective "natural"; it is very different from the perspective often used by 3D software –like graphical spreadsheets- where the perspective is exaggerated for questionable aesthetic reasons.
>
> Since 3D movements are to be performed by way of a 2D pointing device, non-trivial decisions have also to be taken relatively to how a user can drive points in space. He or she must feel "at home" while moving objects within the scene. Most of the pointing devices are 2D devices… For 3D one could think of 3D pointing device; they exist and are still quite expensive; one could also think of user full immersion in a 3D virtual reality environment. To keep technology affordable and widely available, we decided for Cabri 3D to stick with ordinary 2D pointing devices and make use of the old typewriter metaphor: pressing the *Shift* key actually causes a vertical motion of the carriage. In Cabri 3D moving the mouse normally produces a movement of the dragged object in a horizontal plane while pressing the *Shift* key changes this into a movement along the vertical axis. (p. 435)

In my own initial explorations with Cabri 3D I did not find this last metaphor (the typewriter shift key to move vertically in space) an intuitive way of creating 3D figures on the 2D Cabri screen. Not being aware of this design feature (I didn't read the available on-screen help menu for manipulations before exploring!) I struggled for almost an hour trying to create a pyramid but always ending up with a quadrilateral in the represented plane. My intuition was to move the cursor vertically above the plane to create the vertex of the pyramid. Unfortunately, this seemingly vertical motion was interpreted by Cabri 3D as motion in the plane away from me. The constraint I was faced with (and also the designers – see above quote) was how to indicate motion in three dimensions with a 2D pointing device. Perhaps we should be looking to the gaming world for 3D motion controllers such as the Nintendo Wii controller as a possibly affordable 3D motion device for use with 3D dynamic geometry environments?

Scaling Up Integration of Technology: The SimCalc project

In the section on Research Methodologies above, I suggested that experimental-control designs were not appropriate to address most of the questions I have posed in this paper.

However, there is a place (and time) for such research when the technology is at a point in its development and implementation where large-scale, faithful adoption is possible. An implementation of the *SimCalc* simulation tools, together with the *MathWorlds* curriculum has been recently tested with more than a thousand middle school students and their teachers in the state of Texas (Roschelle, Tatar, Schechtman, Hegedus, Hopkins, Knudson & Stroter, 2007). Developed by Jim Kaput and colleagues at the University of Massachussetts-Dartmouth over the past 15 years, the SimCalc software and MathWorlds curriculum have undergone rigorous cycles of development-field testing-revisions. According to Roschelle et al.,

> SimCalc software engages students in linking visual forms (graphs and simulated motions) to linguistic forms (algebraic symbols and narrative stories of motion) in a highly interactive, expressive context. SimCalc curriculum leverages the cognitive potential of the technology to develop multiple, interrelated mathematical fluencies, including both procedural skill and conceptual understanding (p. 2).

In terms of the theoretical frameworks outlined above, the SimCalc software acts as a semiotic mediator, linking several different semiotic systems to develop both procedural skills and conceptual understandings. The results of this extensive implementation of the SimCalc MathWorlds curriculum do, indeed, indicate the cognitive potential of the technology, achieving what has been termed the "gold standard" for experimental research, both in design and effects. A highly statistically significant main effect ($p<0.0001$) was found between control and treatment classrooms on measures of students' conceptions of rate and proportional reasoning after implementation of a three-week unit on rate and proportions. Moreover, the gains for the treatment group were consistent across SES, race and gender groups. Based on these results, the researchers claim the following:

(a) that the SimCalc approach was effective in a wide variety of Texas classrooms,
(b) that teachers successfully used these materials with a modest investment in training, and
(c) that student learning gains were robust despite variation in gender, ethnicity, poverty, and prior achievement. (Roschelle et al., 2007, p. 6)

The researchers make the important point that the gains were accomplished by the treatment students on the more complex items dealing with proportionality and rate, whereas all students made similar gains on the simpler items.

Jim Kaput (1998) pointed out that dynamic, interactive software like SimCalc opens up the *Mathematics of Change & Variation* (MCV) to students who have traditionally been shut out by "the long set of algebraic prerequisites for some kind of formal Calculus, this despite the fact that the bulk of the core curriculum can be regarded as preparation for Calculus" (p. 7). Kaput goes on to state:

> …we can see that while large amounts of curricular capital are invested in teaching numerical, geometric and algebraic ideas and computational techniques in order that the formal symbolic *techniques* of Calculus might be learned, the ways of thinking at the heart of Calculus, including and especially those associated with the Fundamental Theorem, do *not* require those formal algebraic techniques to be usefully learned. Indeed, by approaching the rates-totals connections first with constant and piecewise constant rates (and hence linear and piecewise linear totals), and then gradually building the kinds of variation, we have seen the underlying relations of the Fundamental Theorem become obvious to middle school students. (p. 7)

Beginning with the Dynamic Number tools in elementary school and progressing on to the SimCalc--MathWorlds activities in middle school could be a very powerful combination for students of all ages to engage in the Mathematics of Change and Variation. Coordination of design and curriculum development between these two projects could be very productive.

Examples of New Media for Teaching and Learning

In this section I provide just a few examples of the new media with which I am personally familiar and provide references (where possible) to research that is being conducted or needs to be conducted both within and on these new media. These examples are by no-means exhaustive.

Logo and Robotics

A tremendous quantity of research has been conducted on young children's interactions with the Logo programming language, first popularized by Seymour Papert in his ground-breaking book, *Mindstorms* (Papert, 1980) on young children and technology. Clements and Sarama (2005) provide an overview of this research and the overwhelming evidence that young children do gain insights into spatial relationships, properties of geometric shapes, and linear and angle measurement concepts through their programming activities. Clements and Sarama raise the question "Why not just draw on paper?" rather than struggle to command an on-screen turtle to draw a shape. They answer with the following:

> First, drawing a geometric shape on paper, for example, is for most people a motor procedure. In creating a Logo procedure to draw the shape, however, children must analyze the visual aspects of the shape and their movements in drawing it, thus requiring them to reflect on how the parts of shapes are put together. Writing a series of Logo commands, or a procedure, to draw a shape "allows, or obliges, the child to externalize intuitive expectations. When the intuition is translated into a program it becomes more obtrusive and more accessible to reflection" (Papert, 1980, p. 145) (p. 57)

Clements and Sarama (2005) site many studies from the 1980's through the year 2001 that indicate that this externalization and reflection *does* happen.

The Turtle Geometry of Logo (Papert, 1970) was initially developed as a control language for a physical, dome-shaped robot (dubbed the "turtle"). The expense of the physical device and control mechanisms in the late 1970's and early 1980's made the physical robot turtle prohibitive as a classroom-based learning tool. Mass production of similar control systems with small robotic devices for the toy market, have now made the use of robotics a possibility again in K-12 classrooms. Programming robotic vehicles to travel around obstacle courses, or navigate a specific route, while providing a fun, game-like context, has the potential for rich mathematical learning. I had the good fortune to visit a second grade classroom in Taiwan in May where children were exploring properties of squares, rectangles and triangles by planning a series of Logo-like commands for a physical Lego® robot to trace out the boundaries of these different shapes. During the presentation I shall show video excerpts from that visit that illustrate the points that Clements and Sarama (2005) make concerning the influence on children's communication, collaboration, planning and reflective activities that such programming can have under the guidance of a teacher who has planned a well structured sequence of activities. A group at the National Pingtung University of Education in Taiwan, headed by Ching-Hua Chien, is collecting video data of the Lego-Logo classroom activities with the goal of analyzing both the teaching and learning that is enabled through such activities.

Single & Multi-Player web-based gaming environments

Shaffer et al. (2008) urge the educational research community to look at video and web-based games,

> not because games that are currently available are going to replace schools as we know them any time soon, but because they give a glimpse of how we might create new and more powerful ways to learn in schools, communities, and workplaces—new ways to learn for a new information age. (p. 2)

Shaffer et al. point out that "the virtual worlds of games are powerful because they make it possible to develop *situated understanding*." (p. 5 italics in the original) They make the point that students *will* learn from video games and their involvement in the web-based gaming environments. We need to find out *what* they are learning and *how*. We also need to be concerned about who is creating these games and whether or not they are based on sound theories of learning and socially conscious educational practices. Currently there is a large gulf between educators who try to design fun learning games and gamers who try to add learning in fun games. Neither group has met with much success. What is needed is a genuine collaboration between these two groups. I provide the following examples of multi-player gaming environments from these two camps. The first is a new initiative by NCTM called *Calculation Nation*™: http://calculationnation.nctm.org/, the second is Disney's Club Penguin: http://www.clubpenguin.com/

NCTM's Calculation Nation is a web-based, single or multi-player gaming environment that:

> uses the power of the Web to let students challenge opponents from anywhere in the world. At the same time, students are able to challenge themselves by investigating significant mathematical content and practicing fundamental skills. The element of competition adds an extra layer of excitement (obtained from the web page).

While I did find the games both challenging and sort of fun, I found only 3 players on-line when I logged in (compare this to *Club Penguin*, below). I have played Slam Ball and Prime Time. Both games are mathematically challenging and captivating. They don't have any context or story-line, but I was certainly involved in thinking about the mathematics and trying to beat the computer. Unfortunately, when I tried "Challenging Others" to Slam Ball no one was on-line! *Calculation Nation* is free to join. My on-line name is "ProfJohn" if anyone wants to challenge me to a game of Slam Ball!

Disney's Club Penguin is an on-line, virtual world for young children who interact with each other and the world via their own penguin avatar. More than *20 million* children world-wide currently play in Club Penguin. In the guise of their penguin avatars they can visit different locales and engage in social activities (like dancing, playing hockey or throwing snowballs) and competitive tasks and games. Each locale has different games or tasks in which players can choose to engage. Just one of those games, *Card Jitsu* (a variation on "paper-scissors-rock") purports to attract 24 million players PER MONTH! With such vast numbers of players, we have an amazing data source to investigate the patterns of reasoning and strategies that children develop to progress in the levels of this game.

The original founder of Club Penguin, Lane Merrifield now oversees development of all Disney's online games. A recent report in the *Times* of London (Mostrous, July 17, 2010) indicated that in two months' time (i.e. September, 2010) players in Club Penguin will be rewarded for completing games designed to test their verbal and mathematical skills. According to Merrifield "It's going to take learning within a virtual space to a whole new level." He likens the new Club Penguin to Public Broadcasting's *Sesame Street* with the added attraction of personal interactivity through the on-line virtual world. He is determined to keep things fun: "One of the mandates I gave was that there's no chocolate-covered broccoli. If we're going to do learning, [we have to] recognize that learning can just be chocolate." I believe, that as concerned educators, we have a responsibility to guide the production of such "learning candy" through the application of both sound learning theory and systemic research.

New media for face-to-face collaborative problem solving

While the world-wide-web provides a medium for virtual collaboration, there are new media emerging for real-life collaborative problem solving, both in-school and out-of-school. In the classroom, the new SMART Table offers young children the opportunity to work together to solve virtual puzzles (such as Tangrams) and work collaboratively to generate answers using the multi-touch technology. For example, simple arithmetic computational drill can become an exciting cooperative effort when the answer to 18+9 has to be represented by the correct number of fingers touching the table at the same time – how many pairs of hands are needed? The following YouTube video shows children's use of and reactions to this new technology: http://www.youtube.com/watch?v=ZU-CYeiKmP4&NR=1

New interconnectivity systems such as the TI-Navigator and the SMART response systems offer possibilities for students to send their solutions to a problem to the teacher's computer for classroom display and comparison. These systems can be used to gather data, compare solutions, and investigate different possible solutions to the same problem. They also provide the teacher with real-time access to individual students' solutions so that individual problems can be monitored and dealt with as they occur. Trouche and Hivon (2009) raise the following question concerning classroom connectivity technology: "What are relationships between what we call orchestrations (Trouche 2004; Drijvers & Trouche 2008) – the intentional organization by the teacher of the various tools available in a learning environment, and creativity of the learners who form part in this situation?" (pp. 444-445) Trouche and Hivon (2009) studied the integration of the TI-Navigator system into the French school system at 10th grade. They concluded that work with the system fostered "an emergent real community of practice (Wenger, 1998) in the classroom in which we could distinguish three fundamental aspects, *participation, reification,* and the existence of shared resources." (p. 447) They also reported, however, that the work of the teachers became much more complex as they struggled to manage both students' tools and the collective tool (the calculator network). The researchers intend to undertake a deeper analysis of students' learning processes in such environments when both teachers and students are more used to working in them.

Outside of school, a new kind of collaborative exploration medium is being created using sophisticated digital video and touch technology. Chronis Kynigos, director of the Educational Technology Lab at the University of Athens in Greece is involved in the creation of a new Exploratorium called *Polymechanon* (kinesthetic cooperation). Polymechanon opened its doors to groups of school children just over two years ago. I had the opportunity to visit the canvas-covered adventure playground in Athens in July 2009. I watched in awe as young children excitedly moved their bodies to cooperatively engage in the game situations. I was also thrilled to participate in the games myself. The following YouTube video provides a glimpse of this excitement: http://www.youtube.com/watch?v=d8AJwADKd90. It is narrated by Chronis Kynigos in Greek but has English subtitles. Kynigos concludes with the following:

> In Polymechanon the goal is to give out the message that learning is something absolutely natural. It doesn't need to be tiring, boring or even disconnected from our personal interests and wishes. It could, on the contrary, be a delight or something useful, happening with our friends…or the ones we made when we visit Polymechanon.

What needs to be investigated is the nature of this "natural" learning. *What* are the children learning from these kinesthetic cooperative activities? And *how* are they learning?

Concluding Remarks

Cathy Adams (2006) wrote: "The technological milieu is shaping substantially—insinuating itself, habituating us, and simultaneously reinterpreting—how we act in and perceive the world." (p. 3) My attempt in this paper and presentation has been to indicate, through specific examples, the broad range of questions that this technological milieu raises for mathematics education. I have also attempted to give a glimpse of the range of new technological tools and environments that can have a profound affect on how and what we learn and teach. This is certainly an exciting field of research, involving professionals from many fields and demands a collaborative research agenda. We must be cognizant, however, of the theoretical frameworks that different researchers from different fields employ in their work. Collaboration requires the use of *integrative* frameworks that can take into account the different perspectives of the various disciplines involved. I have offered one such framework in this paper.

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