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Meixia Ding & Kayla Heffernan

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Transferring specialized content knowledge to elementary classrooms: preservice teachers' learning to teach the associative property

Meixia Ding [a] and Kayla Heffernan [b]

[a]Department of Teaching and Learning, Temple University, Philadelphia, PA, USA; [b]Department of Mathematics, University of Pittsburgh at Greensburg, Greensburg, PA, USA

ABSTRACT
This study explores how preservice teachers (PSTs) transfer the intended specialized content knowledge (SCK) to elementary classrooms. Focusing on the case of the associative property of multiplication, we compared three PSTs' SCK during enacted lessons in fourth grade classrooms with their own learning in professional development (PD) settings. Findings revealed the PSTs' successes and challenges in unpacking an example task, especially in areas of making connections between concrete and abstract representations and asking deep questions that target quantitative interactions. Factors that may have supported or hindered PSTs' SCK transfer include the complex nature of teacher knowledge, the PD effort and the outside factors such as the support from textbooks and cooperating teachers. Implications for teacher education and directions for future research are discussed.

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Preservice elementary teachers; knowledge transfer; specialized content knowledge; the associative property of multiplication; case study

1. Introduction

Preservice teachers' (PSTs) learning to teach mathematics has been found to be challenging (e.g. [1–4]). To support PSTs' learning, researchers recently argued for the importance of specialized content knowledge (SCK) for teaching mathematics [5]. This is a type of content knowledge specifically needed for the teaching of mathematics. Although there are studies that have explored PSTs' learning of SCK in teacher education (e.g. [2,6,7]), very few have investigated how PSTs transfer the targeted knowledge to actual classroom settings. One reason for this lack of research may be due to its complexity of conducting studies of this type that demands access to both settings. This study undertakes this endeavour by exploring PSTs' SCK transfer from a teacher education programme to elementary classrooms. By SCK transfer, we refer to the process of putting knowledge into action. As Osterloh and Frey [8] pointed out, the knowledge transfer itself cannot be observed and measured; however, its outcome can be. As such, this study tracks PSTs transform of several key indicators of SCK from a professional development (PD) setting to elementary classroom settings. In particular, we focus on teachers' unpacking of example tasks through representation uses and deep questions.

CONTACT Meixia Ding Meixia.ding@temple.edu

As Ball et al. [5] pointed out, the mathematical task of teaching 'involves an uncanny kind of unpacking of mathematics that is not needed – or even desirable – in settings other than teaching' (p.400). Therefore, teachers 'must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students' (p.400). To aid this unpacking process, representation uses and deep questions are important SCK facets [5], which are also stressed by the cognitive and classroom research [9]. To situate this study, we focus on one fundamental mathematical idea, the associative property of multiplication. This property, along with the commutative and distributive properties, can be initially learned in arithmetic, which will lay a strong foundation for future learning of algebra [10–12]. As such, the targeted topic is also viewed as an early algebra topic [11]. Through the case of PSTs' learning to teach the associative property, we ask: (1) How do PSTs learn to unpack example tasks through representation uses and questioning in PD settings? and (2) How do PSTs unpack example tasks through representation uses and questioning in the elementary classrooms?

2. Review of literature

2.1. Specialized content knowledge (SCK)

SCK is a key component of 'mathematical knowledge for teaching' (MKT, [5]). MKT is developed from Shulman's [13] *Pedagogical Content Knowledge* (PCK) by complementing it with two major components on subject matter knowledge: common content knowledge (CCK) and SCK. Consider, for example, the associative property of multiplication. Knowing that $(3 \times 2) \times 4 = 3 \times (2 \times 4)$ is an instance of the associative property of multiplication that can be considered as CCK, which may hold for many educated adults. However, being able to use specific representations to illustrate this property so that elementary students can make sense of it demands SCK. MKT also contains two main categories of PCK, that is, knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Using the example of the associative property of multiplication, knowing students' common misconceptions of this property is deemed as KCS while having ready strategies to deal with misconceptions belongs to KCT. According to Ball et al. [5], among these four components, SCK is a unique type of content knowledge specifically needed for teaching (as opposed to CCK) and it does not demand knowledge of students and of teaching context (as opposed to KCS and KCT). Due to these merits, Morris et al. [2] argued that SCK is a viable candidate for teacher education and thus should draw increasing attention of teacher education. In fact, Leavy [14] found that obtaining knowledge in one of the four major subcomponents of MKT can motivationally impact learning in the other subcomponents. Thus, by impacting SCK in this study, it is likely that the other components of knowledge will also be influenced.

Although Ball et al. [5] have not provided a definition for SCK, these researchers suggest a list of 'mathematics tasks of teaching' (p.400) that makes this content knowledge special. In the list, the core aspects included teachers' use of examples, representations and deep questions, which are in support of the educational and cognitive literature (e.g. [9]). Additionally, Pashler et al. recommended seven instructional principles as a practice guide for organizing instruction to improve student learning. Among these, recommendations on worked examples, representations and deep questions are mostly relevant to instruction on a single mathematics lesson (the remaining included 'spacing learning over time', using pre- and post-quizzes, etc.). As such, we focus on these three core aspects as a conceptual framework when studying SCK transfer. In what follows, we elaborate on each aspect.

2.1.1. Worked example

For example, Ball et al. [5] suggested using an example to make a specific mathematical point. Thus, teachers' use of examples might serve as a window on teachers' MKT [15]. These assertions were aligned with that of cognitive research where worked examples (problems with solutions given) were found effective in developing students' relevant schema for solving new problems [16–18]. Consequently, teachers' use of worked examples prior to students' own problem-solving was recommended to reduce cognitive load and enhance learning [9]. Past studies on worked examples were mainly conducted in labs by showing students complete solutions. Given that students' learning is not passive, a teacher in a mathematics classroom should engage students in the process of working out an example and making the underlying principles explicit. This process demands teachers' SCK to unpack an example through representation uses and asking deep questions.

2.1.2. Representations

Teachers' representation use is another key component of SCK. Ball et al. [5] stressed 'selecting representations for particular purposes, recognizing what is involved in using a particular representation, and linking representations to underlying ideas and to other representations' (p.400). Other researchers also argued for the use of representations to facilitate students' modelling process [19]. There were multiple representations that may be classified as either concrete or abstract. One may consider a story situation or a picture as more concrete than numerical symbols because the former may activate students' first-hand experiences to aid learning [20]. In fact, the use of multiple representations has been advocated by the mathematical education field [21]. Previous studies also have shown that instruction involving multiple representations enhances mathematical understanding [22,23]. However, research also pointed out that all representations contain limitations, which calls for connection-making among these representations. For instance, concrete representations often carry surface information that may hinder students' seeing the underlying principle, while abstract representations are distant from students' personal experience, which can cause inert knowledge. Therefore, it is important to help students make connections between concrete and abstract representations [9]. Unfortunately, teachers in the United States often introduced multiple representations sequentially without making connections between various representations. This may decrease the potential for supporting student learning [24]. Recent research assertions favoured fading from concrete to abstract, which is also called concreteness-fading. In this sequence, representation 'begins with concrete materials and gradually and explicitly fades toward more abstract ones' ([25], p.10). This technique takes advantage of both concrete and abstract representations and is found to be most effective in supporting both learning and transfer of mathematical concepts [20,26,27]. However, literature shows that teachers and textbooks in the United States often hold a belief that is opposite from this learning sequence [28]. This is because word problems were treated as 'harder' than computation problems and consequently arranged after the computations.

2.1.3. Deep questions

During the process of unpacking an example task, it is also important for teachers to ask deep questions to elicit students' self-explanations of underlying concepts and relationships [29]. Ball et al. [5] suggested 'asking productive mathematical questions, responding to students' "why" questions, and giving or evaluating mathematical explanations' (p.400). Examples of deep/productive questions include why, why is X important, what is the evidence of X, how did X occur, what if, what if not and how does X compare to Y [9]. Without teachers' deep questions, students may not provide deep explanations. In fact, learning effect can be enhanced when students are prompted to self-explain a worked example solution [30]. Therefore, teacher questions, along with other responses to student explanations (e.g. revoicing, orchestrating), have been viewed as indicators of a teacher's knowledge for teaching (e.g. [31]) and key factors of classroom instruction [32,33].

2.2. PSTs' developing and transferring SCK to teach mathematics

Prior studies reported PSTs' difficulties in developing SCK [2,3,6]. For instance, to unpack a decimal task that exemplifies the learning goal, many PSTs could not identify the necessary subcomponents (e.g. relationships between decimal units) of the targeted concept [2]. This result was consistent with Simon and Blume [3] where PSTs who knew the compressed formula (e.g. area = length $\times$ width), lacked the ability to justify why this formula made sense based on the subcomponents and representation uses. In terms of the associative property of multiplication targeted in this study [algebraically, $(a \times b) \times c = a \times (b \times c)$], Ding et al. [34] conducted a survey with PSTs before their taking of the mathematics methods course. It was found that PSTs who knew what the associative property was (CCK) had difficulties making connections between pictures and corresponding number sentences when asked to illustrate the property (SCK). One of the sources of difficulties was related to the PSTs' weak understanding of a sub-concept, the meaning of multiplication (e.g. 3×2 means 3 groups of 2). For example, the PSTs tended to explain the meaning of a single quantity (e.g. 3 tables, 2 plates) rather than the interactions between quantities (e.g. 3 tables of 2 plates). Focusing on single quantities only touched upon surface information, preventing the deep-learning that demands an understanding of the quantitative interactions [35]. In fact, some PSTs did not have a clear understanding of the meaning of multiplication, viewing 3×2 as meaning either '3 groups of 2' or '2 groups of 3'. Although there were different views on the meaning of $a \times b$ as 'a groups of b' or 'b groups of a' or both[1], Wu [36] stressed the importance of consistency in applying the same meaning during teaching to avoid confusion:

> Note that we have implicitly set up a convention in the above definition of multiplication. The product 3×5 could be defined equally well as $3 + 3 + 3 + 3 + 3$, i.e. 3 added to itself five times, but we have chosen to use the other convention instead: 5 added to itself three times. What is important is that, once we adopt this convention, we stay with it throughout the book to avoid confusion. The same remark applies to your teaching in the classroom. (p.28)

We agree with Wu that it is important to apply the same meaning of multiplication during teaching. In fact, we argue that this is an important SCK component because teachers with this type of mathematical knowledge can potentially develop students' mathematical reasoning and sense-making. Ding [6] reported that such SCK component is learnable. With 'spaced learning' [9] over a semester, PSTs who were initially confused by the meaning of multiplication (a groups of b refer to $a \times b$) demonstrated success in consistently applying the meaning to illustrate the property in their end-of-semester test. To what extent such learned SCK may be transferred into elementary classrooms calls for further exploration.

Although there are few studies conducting fine-grained comparison between PSTs' learning in teacher education and their enacted teaching in classroom, prior research does document PSTs' struggles in teaching mathematics. In Borko et al. [1], a PST who had learned how to illustrate the standard algorithm for fraction division through a methods course, could not respond to a student's question about why the procedure of 'inverse and multiply' worked. Even after drawing a concrete picture to justify the procedure, the PST became lost in the process of explanation and told students to simply follow the procedure. The authors attributed the PST's failure in SCK transfer to her inadequate knowledge, along with her poor commitment. In addition, Sullivan [37] reported that PSTs who believed in the importance of using concrete aids did not virtually use them during teaching. These researchers found that PSTs' reliance on textbooks stimulated their short-term goals, which led to rule-focused teaching with the use of concrete aids becoming less urgent. As such, it seems that textbooks may serve as another factor that affected PSTs' transfer of the learned knowledge to teaching practices.

In summary, prior studies either explored PSTs' SCK development in teacher education or their enacted teaching in classroom setting. A detailed comparison of teachers' SCK in both settings is generally lacking, which may otherwise serve as a critical avenue to understand PSTs' successes and challenges in learning to teach. As such, our current study extends prior research to explore PSTs' SCK transfer from a teacher PD setting to elementary classrooms. Specifically, we will compare, between both settings, PSTs' learning to teach the associative property of multiplication through using worked examples, representations and deep questions. It is expected that findings based on this case study can inform the field with ways to better prepare PSTs for teaching early algebra and beyond.

3. Methods

3.1. Participants and project

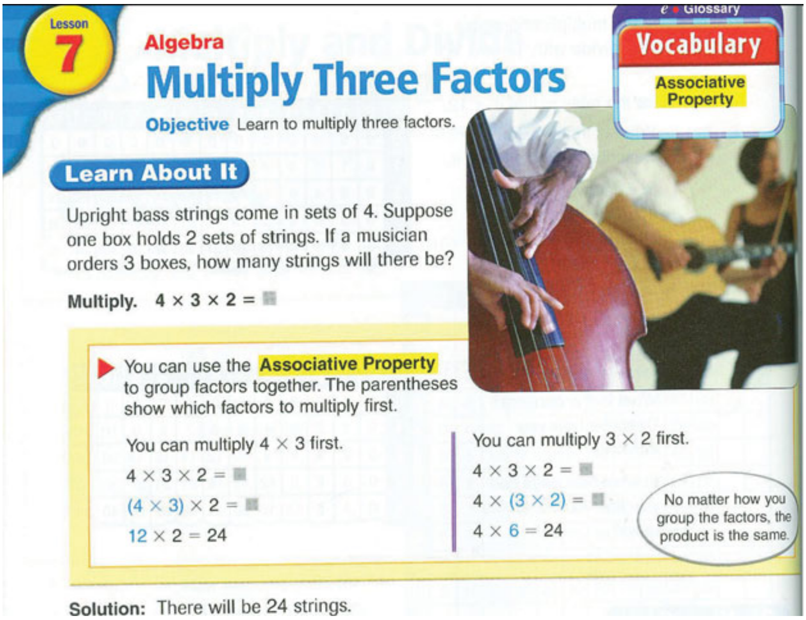
This case study is part of a year-long project aiming to equip PSTs with the necessary SCK for teaching early algebra in elementary school in the U.S. Three participants who registered for a mathematics methods course were recruited. This methods course contained 25 PSTs who were in their third-year at the university. Prior to this methods course, the PSTs took general education courses. This was the first but also last year for them to take the mathematics methods course. The recruiting process took place through an email list provided by the course instructor who was not part of this study. We tried to identify PSTs who had strong interests in learning to teach early algebra, demonstrated willingness to commit to the project and had availability to participate in the project activities. As a result, three Caucasian female PSTs (pseudonyms Anna, Cindy and Kate) were selected. Associated with the methods course, these students would have field experiences in fourth grade classrooms during which they would implement the lesson plans conducted in the methods course. Although the course instructor was not part of this study, she was willing to support the PSTs by allowing them to use the lesson plans and enacted teaching conducted for our project as part of their methods course assignments. In addition, the course instructor acknowledged that her course would not discuss how to teach the associative property of multiplication. Therefore, there would be no specific input from her regarding how to teach this topic. College admission files and knowledge and beliefs surveys indicated that Kate possessed the strongest content knowledge while Anna possessed the weakest; while, in comparison with Kate, Anna and Cindy's beliefs in student learning were more aligned with the reform spirit (e.g. allowing students to construct their own understanding).

For the larger project, PSTs were expected to first attend a 24-h summer training over six inconsecutive days. The discussion was based on Carpenter et al.'s [11] book about early algebra including the concept of equivalence, inverse relations, the basic properties (commutativity, associativity, distributive), variable and expressions and solving equations. Discussions included relevant CCK and SCK for each topic. For instance, we ensured the PST's content knowledge of each concept (e.g. the meaning of the '=', how addition and subtraction are related, etc.). Next, we discussed how each of the concepts may be taught to students through typical example tasks with appropriate use of representations and questioning (we will elaborate the procedures related to the associative property of multiplication in a later section). Relevant research findings were also shared with the PSTs. Specifically, we shared cross-cultural textbook differences in presenting these topics (e.g. [38]) and explained how these research findings may inform our unpacking of the relevant example tasks. After the summer training, PSTs were expected to teach the selected lessons in fourth grade classrooms. Immediate pre- and post-instructional interviews were conducted with the PSTs. To support and enhance PSTs' teaching, we conducted five pre- and post-lessons studies. The current study reports the part relevant to the associative property of multiplication.

3.2. Task analysis

For this study, Lesson 4.7 was selected for teaching from the fourth-grade textbook *Houghton Mifflin Math* [39], a textbook used by the PSTs during their field experience. This was the only lesson that formally introduced the associative property. Although the textbook listed the objective as 'learn to multiply three factors', it highlighted the 'Associative Property' under the 'vocabulary' section and in the example task (see Figure 1). As such, one may reasonably expect a teacher to help students obtain meaningful understanding of the associative property of multiplication. According to Ding [6], this textbook lesson seemed to have two limitations. First, it directly presented the terminology of the associative property of multiplication without any contextual support. In fact, the concrete example task – *Upright bass strings come in sets of 4. Suppose one box holds 2 sets of strings. If a musician orders 3 boxes, how many strings will there be?* – has not been utilized to illustrate the meaning of the associative property (see Figure 1). The second limitation is due to the lack of referents of the suggested solution $4 \times 3 \times 2$. In the two ways of computation, $(4 \times 3) \times 2$ or $4 \times (3 \times 2)$, most steps could not be explained based on the concrete situation (e.g. there was no 4 groups of 3 corresponding to 4×3).

We anticipated the PSTs to utilize the example task but modify and unpack it through modelling and questioning. According to prior research [6,34], instead of directly telling students what this property looks like, a PST may start with a concrete drawing of the problem structure (see Figure 2).



Lesson 7

Algebra

Multiply Three Factors

Objective Learn to multiply three factors.

e Glossary

Vocabulary

Associative Property

Learn About It

Upright bass strings come in sets of 4. Suppose one box holds 2 sets of strings. If a musician orders 3 boxes, how many strings will there be?

Multiply. $4 \times 3 \times 2 = \blacksquare$

You can use the **Associative Property** to group factors together. The parentheses show which factors to multiply first.

You can multiply 4 × 3 first.

$4 \times 3 \times 2 = \blacksquare$

$(4 \times 3) \times 2 = \blacksquare$

$12 \times 2 = 24$

You can multiply 3 × 2 first.

$4 \times 3 \times 2 = \blacksquare$

$4 \times (3 \times 2) = \blacksquare$

$4 \times 6 = 24$

No matter how you group the factors, the product is the same.

Solution: There will be 24 strings.

Figure 1. Textbook presentation of the example task in Lesson 4.7. From *HM MATH*, Student Edition, Grade 4.

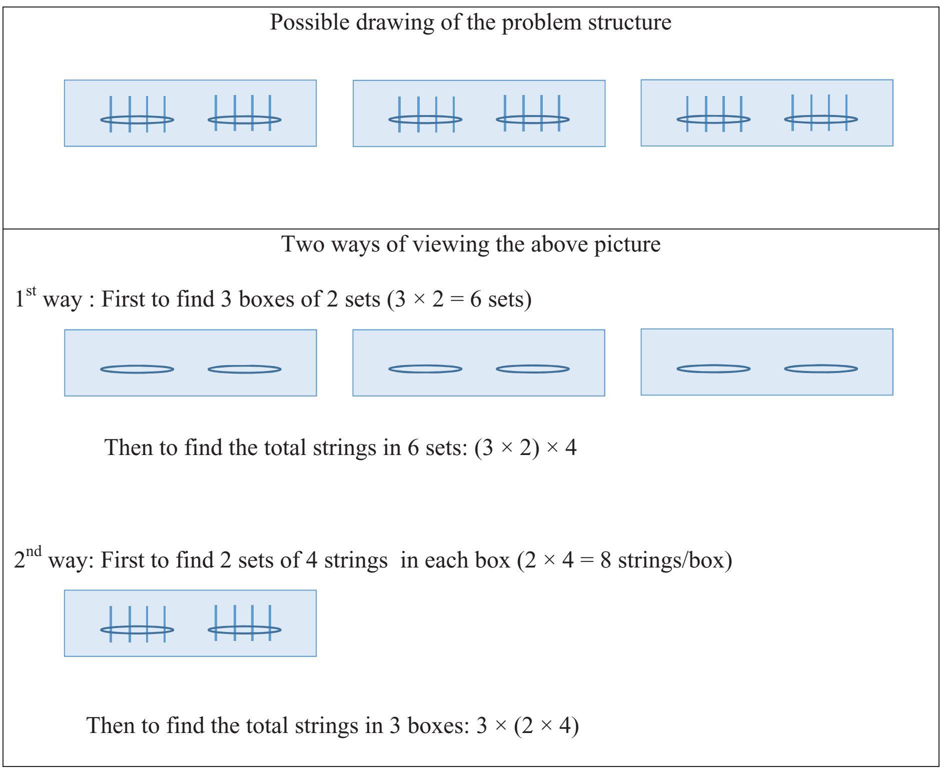


Figure 2. Possible drawing of problem structure for the string problem.

Figure 2 could be viewed in two ways. One may find the total number of sets (3×2) and then the total number of strings, resulting in the first solution $(3 \times 2) \times 4$. Alternatively, one may first find the total number of strings in one box (2×4) and then the total number of strings in three boxes, resulting in the second solution $3 \times (2 \times 4)$. A comparison of the two solutions then generates the equation $(3 \times 2) \times 4 = 3 \times (2 \times 4)$, an instance of the associative property of multiplication. Given that the pictures – in comparison with the number sentences – are relatively more concrete, the representational sequence (from the story context, to drawing, to number sentence) shows concreteness fading [20]. To unpack this example task, a teacher may ask deep questions to elicit student explanations of the meaning of each step (e.g. 'what does 3×2 refer to?'). Such questions target the interaction between quantities, which is the key to deep initial learning [35]. In fact, such questions may prompt students to link the number sentence to its contextual referents (3 boxes of 2 sets/box, or 6 sets), and thus set the learning goal as sense-making beyond answer-seeking.

3.3. Procedures and data analysis

Our PD setting devoted to the associative property of multiplication contained a 1-h summer training and a 2-h pre-lesson study (see Table 1).

Kate missed the summer training due to time conflicts. During the training, we first discussed the relevant CCK including the meaning of multiplication ($a \times b$ refers to a groups of b), the associative property of multiplication and how it is different from the commutative property of multiplication. We ensured that the PSTs were aware of the fact that the commutative property refers to switching numbers around with the product remaining unchanged ($a \times b = b \times a$) while the associative property refers to different regrouping of the factors (not switching the numbers around) with the product remaining unchanged $(a \times b) \times c = a \times (b \times c)$. Next, the instructor introduced research assertions (e.g. [11]) on using concrete contexts to illustrate the basic properties. Based on these discussions, the two PSTs in attendance agreed with the importance of helping students make sense of the associativity. Anna asked how to draw pictures to represent $(3 \times 2) \times 4 = 3 \times (2 \times 4)$, which led to a discussion about how to illustrate the associativity. In addition to the summer workshop, we conducted a 2-h pre-lesson study. All PSTs attended this pre-lesson study. The discussion focused on unpacking the textbook example task (the string problem) to illustrate the associative property of multiplication.

After the PD training, each PST taught a 75-min lesson based on Lesson 4.7 in the elementary textbook, which was observed and videotaped. The PSTs were interviewed for their lesson images before teaching (e.g. Can you walk me through what you plan to do?)

Table 1. Activities of the associative property in PD and elementary classroom settings.

Setting	Activity	Content	Time
PD	Summer training	Research	1 h
	Pre-lesson study	Elementary textbook/lesson plan discussion	2 h
Elementary classroom	Enacted teaching	NA	75 min
	Pre-teaching interview	Lesson images	10–20 min
	Post-teaching interview	Teacher reflections	20–30 min
PD	Post-lesson study	Video-based discussions	1.5 h

and lesson reflections (e.g. Do you think you have accomplished your teaching goal?) after teaching. The interview, which was videotaped, informed the analysis of each lesson and the selection of video-clips for the follow-up post-lesson study. Consequently, during the post-lesson study conducted a few days after the enacted lesson, we watched and reflected upon each video-clip with focused discussions on better use of the worked example, representation and deep questions. All PSTs attended the post-lesson study.

The pre- and post-lesson studies, pre- and post-instructional interviews and the enacted teaching were videotaped and transcribed. Employing the case study method [40], we conducted several rounds of analysis of the video data. We first analysed what the PSTs may have learned from the PD setting, mainly through the discussion of unpacking the string word problem in the pre-lesson study. Features of the PST's representation uses and questioning during this discussion were noticed. Second, we analysed the PSTs' enacted teaching focusing on the SCK components. To obtain a full picture, we first measured the PSTs' efforts in using example tasks by calculating the portion of class time devoted to examples. Since all PSTs taught self-generated examples beyond the string problem, we further analysed the problem structures in order to identify the variations among examples. We also examined the representational sequence (e.g. from concrete to abstract, or from abstract to concrete) used in each example task. Finally, we analysed the PSTs' questioning and tracked their follow-up responses when facing meaningful or non-meaningful student inputs with the goal to identify why PSTs' deep questions may sometimes end with unsatisfactory results. We consider a student's input meaningful if it is explainable based on the story context and does not contain a mathematical mistake (e.g. a student may see 3 boxes of 8 strings and suggest $8 + 8 + 8$); otherwise, non-meaningful. The PSTs' follow-up responses (e.g. accepting or defending a wrong answer or providing direct explanations for students) were also documented. Two independent coders conducted the quantitative coding with reliability exceeding 90%. To enrich the findings, we focused on each PST's enacted teaching of the string problem, focusing on representation uses and questioning. Typical episodes were identified to illustrate the PSTs' teaching moves. Furthermore, we compared and triangulated the PSTs' teaching with the other data sources (e.g. interviews and lessons studies) to understand why they do what they do in classrooms.

4. Results

In this section, we report findings based on two research questions: How do PSTs learn to unpack example tasks through representation uses and questioning in the PD settings and in the elementary classrooms, respectively. Since rich discussions in the PD setting mainly occurred during the pre-lesson study, our finding for the first question was mainly taken from this event. In what follows, we report the findings for each research question, case by case.

4.1. Developing SCK in PD setting

During the pre-lesson study, our discussion focused on how to unpack the textbook example (the string problem) through representation uses and deep questions, which illustrates the intended instructional approaches (e.g. help students make sense of the abstract property through word problem context) agreed by the PSTs in the summer training.

4.1.1. Cindy in PD

The pre-lesson study started with discussing Cindy's PowerPoint that she planned for her upcoming classroom teaching. Using this PowerPoint, we first discussed the string problem. Initially, Cindy stated that she would give students enough time and would expect a solution of $4 \times 3 \times 2$. She would then ask students 'what does the 4 mean? What's the 3? What's the 2?' for her expected solution of $4 \times 3 \times 2$. The other PSTs all agreed with Cindy. As analysed in the Methods–Task Analysis section, the solution of $4 \times 3 \times 2$ does not have a referent in the story situation (see Figure 1). In fact, Cindy's questions focused on individual quantities and may not prompt deep initial learning [35]. Therefore, the instructor challenged the PSTs 'What does 4×3 mean?' and suggested they draw this problem situation. Cindy quickly sketched a picture similar to Figure 2(top). This finding was consistent with Ding et al. [34] in that drawing a diagram is not difficult for PSTs. Based on her drawing, Cindy commented, 'We've got 3 boxes. Each box has 2 sets and each set has 4 strings. How many strings will there be? Now we have a picture. I would write $4 \times 2 \times 3$ ….I wouldn't write $4 \times 3 \times 2$'. Cindy's changing of idea indicated her increased understanding through drawing. It seems the pictorial representation permitted attention to the interaction between individual quantities (e.g. there was an interaction between 4 and 2 but not between 4 and 3). As such, although her solution $4 \times 2 \times 3$ was non-perfect (e.g. there is no '4 groups of 2' but only '2 groups of 4' in the context), it made more sense than the textbook solution. Cindy's increased attention on quantitative interactions was also evident in later discussion of Kate's solution, $3 \times 2 \times 4$. When the instructor asked, 'Show me what 3×2 refers to using this picture', Cindy rephrased it as 'We have 3 boxes of 2 sets. 3 groups of 2', showing her progress toward meaningful interactions.

4.1.2. Anna in PD

During the discussion of Cindy's solution to the string problem, Anna questioned the necessity of stressing meanings, 'I get what you're saying, but I would never put that up there'. This comment is unexpected as she asked for a picture to illustrate the associative property in the summer training. Nevertheless, later in discussions, Anna spontaneously announced that she would also write $4 \times 2 \times 3$:

> I would do $4 \times 2 \times 3$. I would say let's break down the problem. Look at the first sentence. Upright base strings come in groups of 4. I would draw 4 lines. Then I'd say suppose 1 box holds 2 sets of strings. So I would draw a box and draw the sets. I'd say okay, 2 groups of 4. If a musician orders 3 boxes, how many strings will there be. Okay, so I have 3 boxes that look just like the first box. (Anna's suggestion)

As analysed in Cindy's case, although the solution of $4 \times 2 \times 3$ lacked mathematical rigour, it was more meaningful than the textbook solution. However, further discussion revealed Anna's solution was based on the literal rather than structural information in the problem situation (see Episode 1):

Episode 1:
Instructor: How many strings in one box?
Anna: 8.
Instructor: How did you get that?
Anna: 4×2.
Instructor: Why 4×2?

Anna: 2 groups of 4.
Instructor: 2 groups of 4. So why not 2×4?
Anna: Because the problem is written in a different order …Because they say 4 strings and there are 2 sets.

In Episode 1, Anna's generation of the number sentence was not based on the structural information but the literal order in the problem statement (4 strings, 2 sets). This is a common way of thinking held by many PSTs [34]. To challenge Anna, the instructor pointed out that we could rephrase the sentence, 'one box has 2 sets and every set has 4 strings'. This indicates a need to consider 'how many groups of what'. Based on these discussions, the instructor guided the PSTs to modify their proposed solution '$4 \times 2 \times 3$' to the first solution to this problem, $3 \times (2 \times 4)$. That is, first figuring out how many strings in one box (2 groups of 4), then determine the total number of strings in three boxes (3 groups of 8). In addition, Anna demonstrated a robust tendency to focus on individual numbers rather than quantitative interactions. For instance, when later discussing Kate's solution of $3 \times 2 \times 4$, Anna quickly explained the meaning of '3×2' as '3 boxes times 2 sets', which was less deep than Cindy's response of '3 groups of 2'.

4.1.3. Kate in PD

Kate kept quiet most of the time. During the discussion of the solution for the string problem, Kate made a comment, 'Not from drawing, but from algebraically, I was thinking 3 boxes, I would write 3 first. Each box has 2 sets, so 2 next, and each set has 4 strings. So I would write $(3 \times 2) \times 4$'. Kate's suggestion indicated the second solution to the string problem, first figuring out the total number of sets ($3 \times 2 = 6$), and then the total number of strings ($6 \times 4 = 24$). However, Kate generated the above solution based on her algebraic intuition rather than based on the concrete drawing. This indicates a lack of awareness of the connection between concrete and abstract representations. To support the PSTs, the instructor stressed the meaning of each step (e.g. 'how many groups of what'), which served as a model to make connection between the concrete and abstract and to ask deep questions in elementary classrooms. In addition, the instructor suggested asking comparison questions such as 'Comparing the two solutions, what do you find?' – another example of deep questioning. In fact, the instructor pointed out that if we compared the above two solutions, an instance of the associative property can be naturally generated: $3 \times (2 \times 4) = (3 \times 2) \times 4$.

4.1.4. Summary

The learning process during the PD setting contained productive struggles. The PSTs noticed the improper textbook presentation and drew pictures to illustrate the problem structure that was not in the textbook. Although they lacked the awareness of quantitative interactions, with the instructor's consistent request for meaning, the PSTs collectively suggested two different solutions, which contributed to illustrate the associative property of multiplication. With the instructor's continuous support, the PSTs also could explain the number sentences, attending to the quantitative interactions – critical evidence of their SCK gains. Overall, the intended approach – meaningfully solving a word problem in two ways (the numerical solutions of which can be compared to reveal the associative property) – was fully discussed. Note that the instructor explicitly stated that the discussion of the string problem only serves as an example and the PSTs may teach with flexibility.

4.2. Transferring SCK to classroom settings

4.2.1. An overview of the enacted lessons

After the PD setting, each PST taught their planned 75-min lesson to fourth graders. Regarding the worked examples, in addition to the textbook string problem, the three PSTs presented self-created examples: Kit Kat problem and/or the counter problem. The Kit Kat problem stated, 'There are 4 individual Kit Kat bars in a package. If I have 3 packages how many individual bars do I have?'. The counter problem involved using a manipulative consisting of 3 baggies, each with 5 counters, and creating sets by clipping together groups of baggies. Table 2 summarizes the instructional time spent on each example task along with the problem structures of the sub-tasks.

As indicated by Table 2, each PST tried to unpack at least one example task (10+ min). However, the subtasks of the self-created examples appeared to be similar or even easier than the textbook example. Examining the representational sequence among the seven example tasks (see Table 2), four (57%) were from concrete to abstract. These concrete situations often elicited meaningful responses. The remaining three examples were discussed either in an abstract context only ($n = 2$, 28.6%) or went from abstract to concrete ($n = 1$, 14.3%). Consequently, students generated random number sentences with no referents to the word problem situation. Facing students' meaningful or non-meaningful inputs, all PSTs asked questions (R1). To follow-up, the PSTs tended to either accept a wrong answer (R2), defend a wrong answer for students (R3), offer their own explanations (R4) or ignore a student input (R5). Table 3 summarizes the frequency of each PST's responses across example tasks.

Table 2. Worked examples in the enacted lessons.

	Example task	Time spent	Number of subtasks	Structure of each subtask	Representation sequence
Cindy	Kit Kat	4′	3	(a) 3 packages of 4 (b) 9 packages of 4 (*1 box*) (c) 2 boxes of 9 packages of 4	Concrete to abstract
	String	17′	1	(a) 3 boxes of 2 sets of 4 strings	Concrete to abstract
Anna	String	10′	1	(a) 3 boxes of 2 sets of 4 strings	Abstract to concrete
	Kit Kat	16′	3	(a) 3 packages of 4 (b) 9 packages of 4 (*1 box*) (c) 2 boxes of 9 packages of 4	Abstract only
	Counter	10′	2	(a) 3 bags of 5 (b) 2 sets of 3 bags of 5	Concrete to abstract
Kate	Counter	10′	6	(a) 2 bags of 5 (*1 case*) (b) 2 cases of 2 bags of 5 (c) 3 bags of 5 (*1 case*) (d) 2 cases of 3 bags of 5 (e) 6 bags of 5 (*1 case*) (f) 2 cases of 6 bags of 5 counters	Concrete to abstract
	String	5′	1	(a) 3 boxes of 2 sets of 4 strings	Abstract only

Note: 4′ denotes 4 min.

Table 3. Types of teacher responses in the example tasks.

Teacher response	Cindy	Anna	Kate
R1 (question)	44% ($n = 7$)	27% ($n = 7$)	33% ($n = 6$)
R2 (accept)	25% ($n = 4$)	19% ($n = 5$)	33% ($n = 6$)
R3 (defend)	6% ($n = 1$)	23% ($n = 6$)	6% ($n = 1$)
R4 (explain)	19% ($n = 3$)	27% ($n = 7$)	28% ($n = 5$)
R5 (ignore)	6% ($n = 1$)	4% ($n = 1$)	0% ($n = 0$)

In the next sections, we report each teacher's enacted teaching starting with an overview followed by the detailed case of the string problem. In particular, the instructional flow of the string problem in each class is illustrated in Table 4.

4.2.2. Cindy's enacted teaching

Cindy taught two examples (Kit Kat, string), which counted as 28% of her total instruction time (see Table 2). Her representational sequence in both examples was from concrete to abstract. That is, she always started with a picture representing the problem structure that elicited students' meaningful input. Facing students' inputs, Cindy always asked questions (R1). In fact, as indicated by Table 3, Cindy asked the highest proportion of questions. However, her follow-up responses also contained accepting student wrong answers (R2) and explaining for students (R4). Occasionally, she ignored students' meaningful inputs (R5) to proceed with what was planned. In the following, we illustrate her teaching of the string problem.

Cindy spent 17 min on the string problem. Her class started with collectively drawing a picture to represent the string problem, which immediately elicited students' meaningful inputs (see Table 4) as indicated in Episode 3:

Table 4. The case of PSTs' unpacking of the string problem.

	Cindy		**Anna**		**Kate**	
Representation use	Concrete to abstract		Abstract to concrete		Abstract only	
Student initial input	Meaningful	Non-meaningful	Meaningful	Non-meaningful	Meaningful	Non-meaningful
Teacher questions and follow-up responses	R1 → R4 → R1 → R4 → R1 → R1 → R2		R1 → R1 → R3 → R2 → R4 → R4 → R3			R1 → R2 → R1 → R3 → R1 → R2

Episode 3:
T: With this picture, can we pull numbers out for our equation?
S6: 6×4.
T: 6×4. Tell me why. You can come up to our picture if you want to.
S6: (came to the board but inaudible.)
T: Okay, so you took 6 times 4 because there's 6 sets and there are 4 in each set. (To the class) When we're looking at this picture, would that be the right answer?
S3: No.
T: Let's think about it. We have 6 sets and there are 4 in each set. What does that equal?
Ss: 24 ….
S20: But the 3 boxes …
T: Okay. So Derri has 6×4. What's another way we can look at this? … instead of 6, Derri, how can we get three numbers for this problem? What did you multiply? The 3 boxes times 2 sets to get 6, does that sound right?
S6: 1 box of 8 and 2 sets of 4.
T: 1 box of 8 and each set has 4. So 8×3. Is that what you are saying?
S6: I added them.
T: Oh, you added 8+8+8?
S6: 8+4+4+4 … (inaudible).
T: So you would still need one more …to add on 4 and then 4 more. That's why multiplication is helping us out because instead of having this problem 4+4+4+4+4+4, that's just what Derri said, 6×4. It's like using that repeated addition. Alright …
T: So with this problem, our lesson is about multiplying three numbers. …I want you to get out your math notebooks. I want you to write down the three numbers you would multiply together for this problem …

This episode reflects the complexity of classroom teaching due to the interaction between the teacher and multiple student resources. To some degree, Cindy's teaching demonstrates SCK transfer in asking deep questions and using the pictorial representations to aid student understanding. When a student suggested the meaningful solution of 6×4 based on the picture, Cindy asked a deep question – 'Tell me why'. She made a sound teaching move by asking this student to come to the board to explain using the picture. When another student wondered why the '3' representing boxes were not used, Cindy asked the broad question, 'How can we get three numbers for this problem?'. Asking for a number sentence that directly multiplies three numbers was an action to ignore students' existing meaningful input (see Table 4). In fact, Cindy could have asked a specific question based on student input, 'How did you get 6?', which would lead to the first solution $(3 \times 2) \times 4$. Instead, Cindy provided a direct explanation for students. Interestingly, Cindy's direct explanation seemed to be unaccepted by the student. Derri (S6) explained that he saw '1 box of 8 and 2 sets of 4'. He continuously added the 4s to the 8 $(8 + 4 + 4 + 4 + 4)$, to which, Cindy successfully linked back to 6×4.

Although the students initially provided meaningful inputs, Cindy did not fully grasp the student inputs and resorted back to requesting a solution involving three numbers as indicated at the end of the episode. The class suggested $4 \times 3 \times 2$, $3 \times 4 \times 2$, $4 \times 2 \times 3$ and $3 \times 2 \times 4$ and found that all three number sentences arrived at the same answer. In the end, Cindy used $4 \times 3 \times 2$ and directly added parentheses to it in both ways to reveal the associative property of multiplication. This was exactly the textbook approach that Cindy criticized during the pre-lesson study. In other words, the discussion in Cindy's class ended on the non-meaningful side (see Table 4). In fact, Cindy could have grasped the students' inputs to orient student attention toward the first solution '$6 \times 4 = (3 \times 2) \times 4$' and the second solution '$8 + 4 + 4 + 4 + 4 = 8 + 8 + 8 = 3 \times 8 = 3 \times (2 \times 4)$', which could be further compared to reveal the associative property of multiplication, $(3 \times 2) \times 4 = 3 \times (2 \times 4)$.

In the post-instructional interview, Cindy appeared to be satisfied with her lesson, 'They understood how to set up a problem'. She was particularly satisfied with her questions, 'I tried to …say, you know, what does the 2 mean in this one, what does the 4 mean, what does the 3 mean in this one? So they knew exactly what they were doing with those numbers and why they were doing those things'. As previously analysed, asking for the meaning of individual numbers rather than the interactions between numbers lacked depth. It was not until the post-lesson study where the specific video-clips were discussed that Cindy made deep reflections on her lessons, 'I should have said, okay 6, what is 6? Okay, 6 is 3 boxes of 2 sets'. She also agreed that the students' meaningful inputs could be better grasped to form the two anticipated solutions so as to illustrate the associative property of multiplication. Overall, Cindy's enacted teaching demonstrated SCK transfer, especially in using representations. However, her flexibility in responding to student inputs could improve.

4.2.3. Anna's enacted teaching

Anna taught three worked examples (string, Kit Kat, counter). Overall, Anna devoted the highest portion of instructional time to worked examples (48%). In the pre-instructional interview, Anna explained that if students could not get the first example, she would then use subsequent examples to further their understanding. Anna's representation sequence demonstrated an inconsistent style (see Table 2). When teaching the string problem, she first called for an equation, followed by a pictorial illustration (abstract to concrete). In her Kit Kat problem, she only asked for an equation (abstract only). In her counter problem, she started with bags of counters (concrete to abstract). During discussion of these example tasks, Anna did ask many questions (R1); however, when students provided wrong responses, she either accepted the wrong answers (R2) or defended for students (R3). In fact, among the three teachers, Anna tended to defend for students most frequently (e.g. 'The way that he multiplied used the commutative property'). She also provided direct explanations for students (R4). The following demonstrates her enacted teaching of the string problem.

After writing the key words of the string problem on the board, Anna requested equations from students without concrete aids (see Table 4). Students provided non-meaningful responses that cannot be explained based on the story situation ($4 \times 2 = 8$, $8 \times 3 = 24$). Anna responded with a deep question but rephrased it to focus on surface information:

Episode 2:
T: Why did you do $4 \times 2 = 8$ first?
S8: Because I had to get the answer to solve the missing number on the equation.
T: What does the number 4 represent? What is 4, 4 what?
S8: Four strings.
T: Four (underlines '4' on board) bass strings.

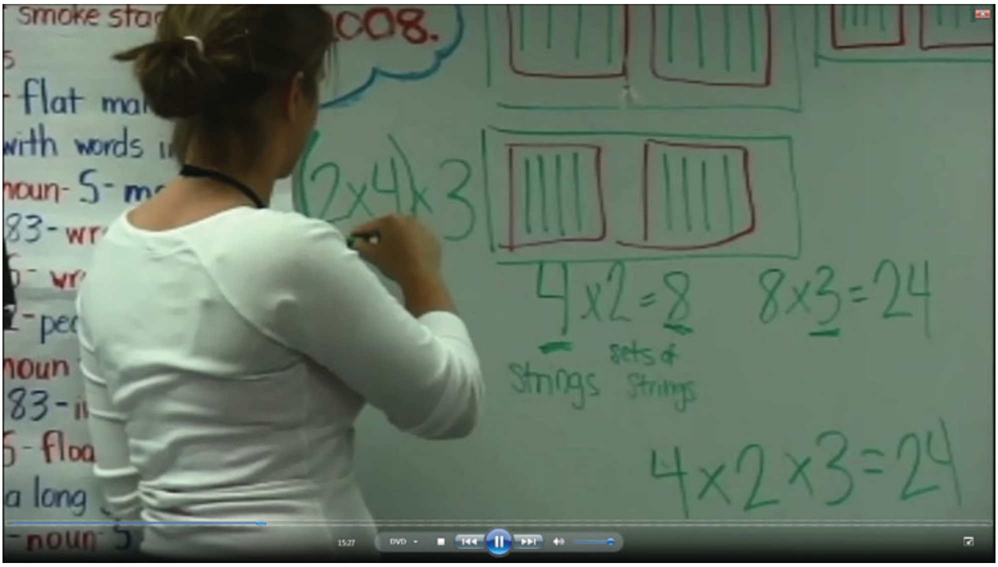


Figure 3. Anna's unpacking of the string problem.

T: Okay, what's your 2?
S8: Times 2 is in one box. It is holding 2 sets.
T: So because you have 2 sets of 4, right? You put 4 × 2, right? Good, that's right.
(Continue to discuss the meaning of '3' until the class reach the answer of 24).
T: Good, let me show you something. I'm going to … draw this picture. (Drawing on board, see Figure 3) … So if you have 4 strings in one set and you have 2 sets, so if this was an array, we would say we have 2 groups of 4, right? Is that how you would say that? Because you can see these 2 groups, right? So we have 2 groups of 4, so I would write it 2 × 4, right? And then I have 3 whole boxes, with each group in it. So I would do that '× 3'. And if you were using associative property, I would say I would do 2 × 4, right? … I think it makes sense to do (2 × 4) × 3. Does anybody have any questions about that?

In Episode 2, Anna's initial question 'Why did you do 4 × 2 = 8 first?' requested an explanation of quantitative interactions; however, when this question was quickly rephrased to focus on individual numbers (e.g. 'What does the number 4 represent? What is your 2?'), classroom conversation remained centred on surface information. Asking for the meaning of individual numbers would not produce a conflict between the concrete (2 groups of 4) and abstract representations (4 × 2). As a result, Anna accepted and even defended students' answers ('So because you have 2 sets of 4, right? You put 4 × 2, right? Good, that's right', see Table 4). These teaching moves were consistent with the pre-lesson study where Anna was reluctant with stressing the meaning. Surprisingly, Anna went further to draw pictures during which she attempted to stress the meaning of multiplication (e.g. 'we have 2 groups of 4, so I would write it 2 × 4'). This teaching move indicates Anna's significant effort to transfer the intended SCK into elementary classroom. However, given that she first accepted and defended students' wrong responses but then tried to stress meaning, inconsistency in instruction might have caused students' confusion. In addition, while attempting to stress meaning, Anna was only correct in the first but not the second step (should be 3 groups of 8). This again conveyed inconsistent messages to students. In fact, her mention of the associative property was out of place. As observed in later discussion, some students stated that they did not understand, further causing frustration for Anna. In the post-interview, Anna complained, 'it was frustrating because they [students] didn't understand anything I was saying …they weren't paying attention, they didn't care'. Anna's frustration drew her back to the skepticism on stressing meaning, which also brought out other concerns:

> I think that meaning is stupid, because yeah we want to teach meaning but … when you're [going to] give them a test, like the worksheet that I gave them, that worksheet that they did had no meaning on it. It was all numbers, all procedure … So, when I try to teach them the meaning, they're going, "just tell me the procedure, so I can memorize it, so I can do well on my test." That's all they care about. (Anna's post-interview)

Anna's complaint was evident in the existing textbook worksheets that simply asked students to multiply three numbers in different ways, which may not necessarily involve the associative property of multiplication. Anna's complaint was also reflected by her cooperating teacher's evaluation on her lesson, which seemed to weaken Anna's reflections. Anna mentioned in the post-lesson study that the cooperating teacher commented, 'I don't know what you feel so badly about …they totally got it! They just took the math test the other day and they did fine'.

In some sense, Anna's reflections contained truth and it is always good to consider student motivation; however, her reflections focused mainly on external factors (e.g. students, assessments) rather than internal factors (e.g. teacher knowledge). Indeed, the deficiency of her own SCK was confirmed through the post-lesson study. When we discussed her own video-clip that contained $(2 \times 4) \times 3$, Anna did not recognize the mistake in the second step. It was not until our later discussion of Kate's lesson that Anna admitted she finally saw this a lot clearer.

4.2.4. Kate's enacted teaching

Kate taught two worked examples (counter, string), which together took 20% of her class teaching time (Table 2). Note that although the counter problem took 10 min, this problem contained six repetitive subtasks (see Table 2). As such, Kate went through each subtask quickly. Like Anna's representation sequences, Kate started her counter problem with bags of counters (concrete to abstract) while her string problem involved abstract equations only. During the discussion, when students provided an incorrect response that was not supported by the modelling perspective, Kate tended to simply accept them without further discussions (R2). Kate did this most often (e.g. 'So you did, the four and the three equals twelve, times two'). Similar to the other PSTs, she also provided direct explanations for the students (R4, see Table 3). The following presents Kate's teaching of the string problem.

Kate's class spent 5 min on the string problem, with the discussion remaining abstract (see Table 4). The class started with selecting the key quantities from the word problem. Next, students suggested an equation that did not have any reference to the story situation. Further, various ways to find the answer for this equation were discussed, which were mistakenly linked to the associative property of multiplication.

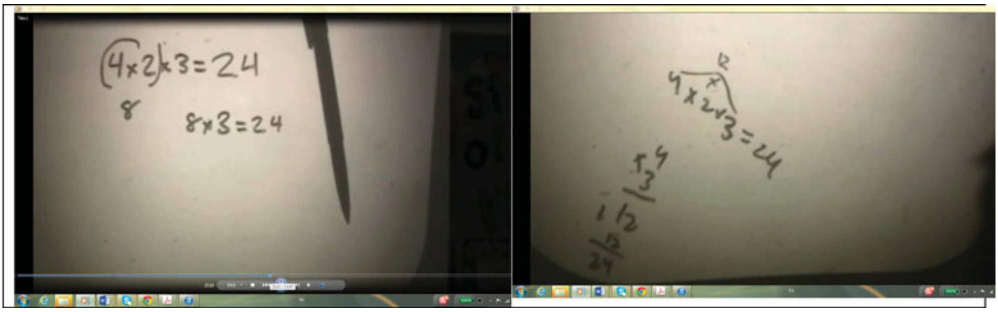


Figure 4. A wrong interpretation of the associative property of multiplication in Kate's class.

Episode 4:

T: First, let's just as a class, what are some of the important pieces of information that you picked up on? Katie.

S13: Sets of 4. … (Kate's student picked 4 strings, 2 sets and 3 boxes; Kate wrote 'Each set = 4; 2 sets = 1 box; 3 boxes = ordered' on the overhead).

T: So our equation, Tyler, do you want to come and write the equation that you got.

S4: [Writes the equation, $(4 \times 2) \times 3$, see Figure 4, left].

T: Tyler, why did you decide to group four and two?

S4: Because four and two equals 8 and I know how to times 8's and it's easier.

T: So it's easier. Anyone else do it a different way? Nicole, come to show us how you did it.

S9: (Goes up to overhead) This is how I did it, 4 times 2 times 3 equals, $4 \times 3 = 12$ (draw a line to link 4 and 3), 12 plus 12, I know that because of the 2, equals 24 (see Figure 4, right).

T: So you did, the four and the three equals twelve, times two. Because you took it two times right? So that is a different way of doing it and that just shows us again, what does that show us?

S9: That shows us the associative property.

T: The associative property of multiplication. Good job.

In the above episode, without any concrete support, Kate's class produced a number sentence $(4 \times 2) \times 3$ that did not reflect the problem structure and was corrected in our pre-lesson study. The follow-up discussion on finding the answer was limited to number manipulation. Although Kate asked students to explain why they did what they do, students' responses were only related to which two numbers were easy to multiply. In the end, Kate guided the class to compare the two different solutions $4 \times 3 \times 2 = (4 \times 3) \times 2 = (4 \times 2) \times 3$ and misinterpreted this as an instance of the associative property. Kate's cooperating teacher re-emphasized this wrong interpretation at the end of the class. Such a misinterpretation might have supported Kate's confidence in teaching as indicated by her post-instructional interview. Overall, in comparison with Anna and Cindy's teaching, Kate's lesson appeared smoother because she generally accepted all students' answers without stressing any meaning of multiplication. In the post-lesson study, Kate explained that she decided not to spend too much time on the meaning of each step because 'they all got it'.

In summary, Anna and Cindy made a greater effort in transferring the intended SCK into enacted classrooms; however, students' input brought challenges. As such, both teachers experienced some frustration. In contrast, Kate aimed to stress procedures and did not transfer the intended SCK into classrooms. Consequently, she was satisfied with her enacted lesson.

5. Discussion

5.1. Successes and challenges in SCK transfer

The current study extends prior studies on SCK [2,6,34] and explores the process of PSTs' SCK transfer from teacher education to elementary classroom. Such an endeavour is rare in the literature likely due to its complexity in aligning the PD effort with the corresponding classroom teaching. Regardless of the complexity, a close inspection of such a transfer process and factors that may support or hinder this process is much needed if we want to help PSTs successfully obtain the necessary SCK for further teaching. Our exploration focused on a unique case, the associative property of multiplication, which is a fundamental mathematical idea emphasized by the Common Core [12] but overlooked in educational research. It is expected that our findings will inform the field about PSTs' successes and challenges in SCK transfer and thus better prepare PSTs for future teaching.

The targeted SCK components in this study were unpacking example tasks, mainly through the representational uses and questioning. Teachers' abilities to select and use examples serve as a window on their MKT [15]. In this study, each PST did spend time unpacking at least one worked example in the enacted lesson; however, discussions focused mainly on solving the word problem itself and then applying the associative property to find the answer. This was inconsistent with the purpose of illustrating the basic property through solving the example task. In this sense, the example task was not treated as a case of a principle [16]. Furthermore, all PSTs created their own example tasks in addition to the textbook example. However, the PSTs' self-created tasks followed the exact form of the textbook examples. Simple repetitiveness of example tasks without variation does not promote students' encoding of the key principle [41].

The PSTs were expected to unpack an example task through connecting concrete and abstract representations and asking deep questions to elicit student explanations [9]. These PSTs demonstrated partial success in transferring both subcomponents of SCK. For example, the PSTs either drew appropriate pictures or used manipulatives during the discussion of example tasks. These drawings and manipulatives served as concrete representations to help connect the problem situation with the abstract number sentences. When concrete representations were provided, some students offered meaningful inputs as part of the class conversation. In addition, at least two PSTs attempted to ask deep questions to help students understand the meaning of abstract symbols and refer them to the concrete representations. The PSTs' success in this regard indicates that with appropriate training, PSTs have a potential to transfer the intended SCK from PD to classroom settings.

Our findings also reveal challenges in the PSTs' representation uses and deep questioning because these SCK components were only transferred to the initial but not later phases of instruction, which is similar to Zevenbergen's [4] finding. With regard to representations, although PSTs in the PD setting agreed upon the importance of using concrete representations for modelling, concrete representations in the enacted lessons were not sufficiently used for mathematical modelling and reasoning. For instance, Cindy expected students to give an equation immediately following the construction of a picture. This indicates a purpose of answer-seeking, allowing students to multiply the numbers any way. If the purpose is to illustrate the abstract idea through modelling, PSTs must guide students to reason upon the drawing in different ways to generate different expressions that can together illustrate the associative property [6]. In fact, PSTs who did lead class discussion on concrete representations failed to pursue deep conversations. For instance, while we stressed meaningful interpretations (how many groups of what) in the PD setting, the PSTs in the enacted teaching resorted to asking the meanings of individual quantities rather than quantitative interactions. In some occasions, they did stress quantitative relations; yet, they were only successful in the first but not the second step. This was again similar to the findings of previous studies [4,6,34].

Teacher questioning demonstrated similar challenges. In most occasions, the PSTs asked an initial 'why' question without following-up with deep prompts. Instead, their follow-up teaching moves – accepting and defending wrong answers, providing own direct explanations and ignoring meaningful student inputs – all risk the abandoning of meaningful teacher–student interactions. Thompson and Thompson [33] emphasized that teachers must 'be sensitive to children's thinking during instruction and shape their instructional actions accordingly' (p.279). Classroom interaction may delineate from facilitating effective learning when PSTs give up the follow-up questions too quickly. In fact, asking deep follow-up questions appears to be difficult even for in-service teachers [42], which should draw attention of teacher education. In fact, although the PSTs asked the why questions, they only anticipated numerical but not relational responses. Focusing on numerical rather than relational calculation [43] is a common issue in current classrooms, which causes the lack of deep initial learning [35]. Overall, the above challenges in PSTs' representation uses and questioning may partially explain why the PSTs lacked the abilities to unpack an example task to illustrate the underlying principle.

5.2. Factors that may support or hinder PSTs' SCK transfer

Despite imperfections in carrying out lessons, the degree to which the PSTs transferred SCK to classrooms appears to associate with different learning gains. In the PD setting, Cindy demonstrated the most understanding of the intended SCK. Consequently, the SCK components were most visible in her enacted teaching. Regardless of the difference, one common factor that affects PSTs' SCK transfer may relate to the complex nature of teacher knowledge demanded for actual teaching. SCK is a special type of content knowledge, not directly dependent on the knowledge of students and teaching [5]. However, when the PSTs try to transfer SCK from the PD setting to elementary classrooms, the complexity of the teaching context and student thinking seems to bring challenges. For instance, while the PSTs in this study could explain the meanings of each step during the PD setting, when elementary students in the enacted lesson presented unexpected solutions, the PSTs failed to ask deep questions to orient students' attention toward meaningful interpretation. This indicates that SCK alone might not be sufficient for pursuing SCK transfer. Perhaps, if PSTs could have stronger knowledge about content, students and the teaching context (KCS and KCT), they may be able to better capture students' thinking and better exercise their SCK in the classroom context. PSTs' lack of KCS and KCT may be related to their own weak CCK because these PSTs themselves possessed similar confusion, which might have blinded their ability to see students' mistakes. The above findings reveal that although there is a possibility to transfer the targeted SCK into classrooms, other components of teacher knowledge play a role [7,14], making the SCK transfer process more challenging. Future studies might explore how these knowledge components may interact with each other during the process of SCK transfer.

Another factor that affects PSTs' SCK transfer may be related to the PD setting. The PD activities in this study only accumulated to 3 h, mainly through a one-time pre-lesson study. This was different from Ding [6] where PSTs had reoccurring opportunities to learn the targeted SCK. Perhaps, the challenges in our study echo the importance of spaced learning [9], which may allow the solidification of PSTs' SCK and better transfer of it to classrooms. In reality, many mathematics methods courses may at most afford one 2–3 h meeting (like this study) to discuss a specific topic like the associative property of multiplication. Our findings, therefore, call for innovative ways to support PSTs' learning in teacher education.

Findings in this study also revealed outside factors beyond teachers' own knowledge and the PD effort. First, the textbook could have better presented the example task in mathematically and pedagogically meaningful ways, with worksheets that go beyond assessing procedural knowledge. Sullivan [37] found that textbooks largely influence teaching, and PSTs' reliance on the text leads to emphasis on short-term goals, which further leads to teaching practices that are procedural and rule-focused. As such, textbook designers may revise the presentation of example tasks and provide key points in representation uses and questioning to support teacher learning [44]. Second, cooperating teachers' knowledge and beliefs could be strengthened to better support PSTs' learning. Given that cooperating teachers are a key factor that directly affects PSTs' growth, teacher education may consider how to provide PD opportunities for cooperating teachers along with PSTs. With a more supportive learning environment, there is greater likelihood that PSTs can be equipped with the necessary SCK and transfer the SCK into elementary classrooms to teach early algebra and beyond.

Note

1. This debate goes beyond the scope of this study. We agree with Wu that regardless of the interpretation, a teacher should use the chosen interpretation consistently to conduct mathematical reasoning. This study chose the first interpretation, $a \times b$ refers to 'a groups of b', because this definition was used by the elementary mathematics textbook series when it initially defined multiplication. Indeed, this is a more popular interpretation in the U.S. However, given that the focus of this study is on SCK, we focus on making connection between concrete representation (e.g. 3 boxes of 2 sets/box) and the corresponding abstract representation (e.g. 3×2).

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ORCID

Meixia Ding http://orcid.org/0000-0002-5820-0684

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