Diagnosing Teachers' Multiplicative Reasoning Attributes

Introduction

This document explains the mathematical content of the two *Diagnosing Teachers' Multiplicative Reasoning* (DTMR) assessments for middle grades teachers. The first assessment is intended to measure aspects of multiplicative reasoning critical for multiplication and division of fractions; the second assessment is intended to measure core aspects of proportional reasoning. We are particularly interested in teachers' capacities to use problem situations and drawn models to develop fraction arithmetic and proportional reasoning with their students. Thus, both tests emphasize reasoning with quantities such as lengths, areas, and volumes, not computation procedures.

The DTMR assessments are designed to be used with Diagnostic Classification Models (DCMs), an emerging family of psychometric models. Whereas traditional Item Response Theory (IRT) models locate examinees along a continuous scale of latent ability, DCMs classify examinees into latent groups. The classifications are based on multiple categorical latent variables, termed "attributes." When using DCMs, test developers specify a set of attributes and use those attributes to design test items. Items can require reasoning with just one attribute or with combinations of attributes. Developers construct full test forms so that each attribute is required by several items. DCMs use an examinee's performance across all items that require a specific attribute to measure whether or not the examinee is a "master" of that attribute. The models then produce a "profile" for each examinee consisting of probabilities that the examinee is or is not a master of each attribute. A test based on k attributes creates 2^k patterns of attribute mastery, and each pattern defines a group into which examinees can be classified. Because DCMs estimate proficiency along multiple constituent dimensions of a given domain, they hold promise for measuring knowledge at a finer grain-size than has been achieved with IRT models.

A main challenge for the DTMR project has been to identify workable attributes for fraction arithmetic and proportional reasoning. We sought attributes that (a) were grounded in the mathematics education research literature on fractions and proportional reasoning; (b) could be reliably measured with machine scoreable paper and pencil assessments; and (c) separated teachers so that the DCMs could generate distinct profiles. We encountered challenges with each of these requirements. First, the mathematics education research literature contains many findings about fine-grained understandings for which researchers have relied on nuances of language, gesture, and sequences of inscription that are not preserved in written responses to test items. Second, multiple-choice items are the most common format for machine scoreable items but can provide weaker evidence for understanding than constructed response items if the choices scaffold reasoning about the targeted content. Third, there are gaps in the relevant mathematics education literature that include a dearth of research on teachers' partitioning and proportional reasoning.

We responded to the challenges mentioned above in several ways. First, the attributes described herein combine clusters of more fine-grained understandings (sub-attributes) reported in the mathematics education research literature. We clustered sub-attributes when we could not write items that reliably discriminated among them. This occurred when teachers could apply any one of several related understandings to answer items correctly. Second, we developed a pool of items that include multiple-choice items, constructed response items that are machine scoreable, and a small number of constructed response items that have to be scored by hand. Third, we conducted interviews to gain insight into teachers' reasoning on topics underrepresented in the research literature.

We arrived at the attributes described herein through several cycles of trial and refinement. These cycles included Item Development interviews and Item Response interviews conducted with in-service middle grades teachers in California, Georgia, North Carolina, and Massachusetts. The Item Development interviews were clinical interviews (Ginsburg, 1997) built around constructed response tasks. The goal of these interviews was to investigate the resources teachers engaged when reasoning about fractions and proportions in problem situations and through drawn models. We used the resulting data to inform our choice of attributes and to determine within those attributes where teachers encountered difficulties. We used information about teachers' difficulties to write items that could separate teachers as masters and non-masters of our chosen attributes. The Item Response interviews were conducted after teachers had answered draft versions of test items. During these interviews we asked teachers why they answered the items in the ways they did. Data from Item Response interviews led us to refine both our items and our attributes when we could not infer intended attributes from teachers' responses. Our final attributes are based both on mathematics education research and on what we could reliably measure with written items.

We used the sub-attributes to ensure that parent attributes were instantiated in a variety of contexts across items on the tests. A teacher who used an attribute appropriately in one situation, might not in other situation where the attribute could also be employed productively. Because our items are just a sample of situations in which the attributes could be employed appropriately, we interpret the term "master" conservatively to mean that a teacher used an attribute appropriately across the situations presented in the items and the term "non-master" to mean that the teacher did not. When considering the sampling issue, we consistently used forms of representation found in curricular materials (e.g., number lines, area models, blocks, etc,.) and/or contexts that teachers encounter routinely (e.g., talking to other teachers and to students) to provide contexts in which teachers with good command of the attributes would likely recognize their relevance.

We recognize that our attributes are not the only ones that could be used and that they do not highlight all important aspects of reasoning about fractions and proportions. Nevertheless, we think that a teacher who is proficient with each of our chosen attributes is likely to have a strong understanding of how to reason about rational numbers in terms of multiplicative relationships among quantities. In what follows, we present the fractions attributes and cited work followed by the proportional reasoning attributes and cited work. Each set of attributes was authored by a different group indicated in the footer.

Supported by the National Science Foundation under Grant No 0903411. The opinions expressed are those of the authors and do not necessarily reflect the views of NSF. Correspondence should be addressed to Andrew Izsák, Department of Mathematics and Science Education, The University of Georgia, 105 Aderhold Hall, Athens, GA 30602. izsak@uga.edu.

References:

Ginsburg, H. (1997). *Entering the child's mind*. Cambridge, UK; Cambridge University Press.

DTMR Attributes for Fractions

Andrew Izsák, Erik Jacobson, & Joanne Lobato

Overview

Table 1 summarizes the four fractions attributes and sub-attributes that contribute to them. We use the subcategories as a tool for insuring that we assess each attribute in a variety of contexts. For each attribute and sub-attribute we provide a general description and examples that illustrate reasoning with that attribute. We also provide references to document how each attribute is grounded in the research literature.

Attribute	Sub-Attributes
Referent Units	Norming
	Referent Units for Multiplication
	Referent Units for Division
Partitioning	Simple Partitioning
	Partitioning in Stages
	Partitioning Using Common Denominator
	Partitioning Using Common Numerator
Iterating	Unit Fractions
Appropriateness	Identifying Multiplication Situations
	Identifying Quotitive Division Situations
	Identifying Partitive Division Situations
Multiplicative Comparisons	(No Sub-Attributes)

Those familiar with the literature know that researchers have decomposed rational number into a set of related but distinct subconstructs (Kieren, 1988, 1993). Researchers involved in the influential Rational Number Project (as summarized in Behr, Harel, Post, & Lesh, 1992) based much of their work on four subconstructs suggested by Kieren: (a) quotient, (b) measure, (c) ratio number, and (d) multiplicative operator. Behr, Wachsmuth, Post, and Lesh (1984) added a fifth subconstruct—part-whole relationships. We do not use these subconstructs as attributes, but they are present in the DTMR items.

Attribute 1: Referent Units

The *Referent Units* attribute covers aspects of reasoning with units that are required when numbers are embedded in problem situations. Teachers who are proficient with numeric algorithms for computing are not necessarily adept at identifying appropriate referent units. We use two related meanings for the term *unit*. The first meaning has to do with a *standard for* *measurement*, and we will sometimes use the term *one whole* as a synonym for this meaning. The standard could be conventional (e.g., 1 inch, 1 square foot, 1 liter, 1 second, 1 degree Celsius, etc.), but a line segment or rectangle provided in a diagram could also establish the standard for measurement. The second meaning refers to *a part* that is either contained in a standard for measurement or contains a standard for measurement. For instance, one might take 1/2 inch or 2 liters as units. The second meaning for unit arises frequently in multiplication and division situations. We examine three aspects of reasoning with units: norming, attending to referent units in multiplication situations, and attending to referent units in division situations.

Attribute 1a. Norming

The term *norming* refers to the establishment of standard units for measurement. (See Lamon, 1994, 2007 p. 644, for discussions of norming.) We emphasize two cases. The first case includes (a) choosing a standard unit from alternate choices and (b) looking at a given problem situation in more than one way based on different choices for the standard unit. The second case includes changing the standard unit as one reasons through a problem. This is sometimes referred to as renorming.

Example 1. Choosing a standard unit for measurement from alternate choices
When using base-10 blocks (see below) to represent numbers, one has to choose whether a block, a flat, a rod, or a cube will serve as the standard unit, or the one whole. When reasoning about whole numbers, one might let the cube (the smallest shape) serve as the standard unit. When reasoning about decimals, however, one might let different shapes serve as the one whole. To represent 6.54, one could let the flat represent 1, the rod represent one tenth, and the cube represent one hundredth (6 flats, 5 rods, and 4 cubes). Alternatively, one could let the block represent 1, the flat represent one tenth, and the rod represent one hundredth (6 blocks, 5 flats, and 4 rods). Sometimes teachers lack flexibility in choosing the standard unit. For instance, when interpreting base-10 blocks, they might think that only the cube can represent 1.

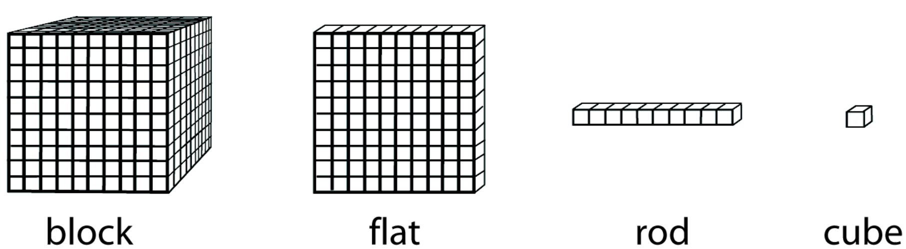
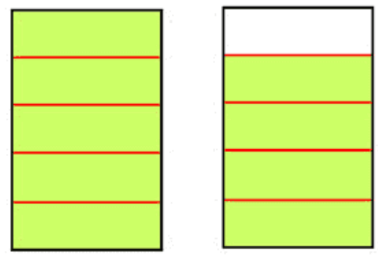


Figure 1. Base-10 blocks.

Example 2. Changing the standard unit for measurement (Renorming)
The following problem requires making two choices for the standard unit for measurement:

> Sam and Morgan are comparing the amount of liquid in their beakers as shown in the diagram below. Sam claims that Morgan has 20% less than she has. Morgan claims that Sam has 25% more than she has. Who is right? (InterMath, http://intermath.coe.uga.edu)



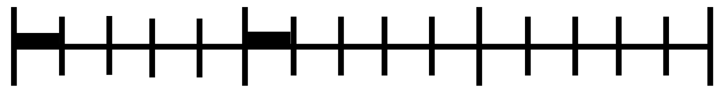
To see that both Sam and Morgan can be right requires first using Sam's amount of liquid as the standard unit and then using Morgan's amount of liquid as the standard unit. Some teachers are not facile at such renorming.

Comments/Boundaries
Norming is closely related to the idea of unitizing, which Lamon (2007) defines as "the process of mentally chunking or restructuring a given quantity into familiar or manageable or conveniently sized pieces in order to operate with that quantity" (p. 630). Unitizing is a major accomplishment for elementary school students (e.g., Olive & Lobato, 2008; Steffe, 1988).

Attribute 1b: Referent Units for Multiplication

As one moves from additive to multiplicative situations, referent units for numbers become more complex. If A, B, and C are values for quantities in some problem situation, then for the equation A + B = C to make sense A, B, and C must refer to the same units. In contrast, in the equation A x B = C each value refers to a different unit. (See Schwartz, 1988, for one discussion of multiplication as a referent-transforming operation.) Although most teachers can use algorithms to calculate the correct product of two fractions or two decimals, several studies (e.g., Armstrong & Bezuk, 1995; Eisenhart et al., 1993; Izsák, 2008; Sowder, Philipp, Armstrong, & Schappelle, 1998) have reported constraints on inservice and preservice teachers' performance when using drawings to explain such products. Some teachers can reason about referent units for multiplication to a point by relying on an association between the word "of" and multiplication, but teachers can still have difficulty identifying appropriate referent units in multiplication situations.

Example 1. Distinguishing part-of-a-part from part-of-a-whole
Izsák (2008) reported a case in which a teacher and her students had trouble understanding one another, at least in part, because they stumbled over referent units for different terms in a fraction multiplication problem. Ms. Archer introduced her sixth-grade students to fraction multiplication with a number line that showed a solution to 1/5 of 2/3 (see Figure 2). The drawing came from the teacher's edition of the Bits and Pieces II unit in *Connected Mathematics 2* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2006). It showed the interval from 0 to 1 subdivided into thirds. Each third was further subdivided into 5 parts. Two parts were shaded as shown. Using the diagram to determine the answer to 1/5 of 2/3 requires combining two interpretations of one bold segment: It is 1/5 of the 1/3 length, and it is 1/15 of the one whole. Thus, in an appropriate interpretation of Figure 2, the referent unit for 1/3 and 1/15 is the one whole, but the referent unit for 1/5 is 1/3.



0 1/3 2/3 1

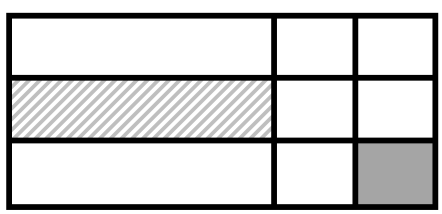
Figure 2. A solution to 1/5 of 2/3 on the number line.

One student asked Ms. Archer why she talked about 1/5 as 1/15. Apparently thinking of equivalent fractions, the student thought 1/5 was equal to 3/15. He did not seem to realize that the referent unit for 1/5 was 1/3. Ms. Archer had trouble responding to her student, at least in part, because she herself did not maintain explicit and appropriate referent units for each fraction in the problem. She discussed 3/15 sometimes as if the 15ths referred to parts of the whole and sometimes as if they referred to parts of 1/3. In a subsequent interview during which she reviewed a video recording of this lesson excerpt, she still did not discuss explicitly the different referent units for each fraction in the problem.

Example 2. Reasoning when the whole is not present visually

We have found through interviews conducted as part of the DTMR project that teachers can struggle to reason about parts of parts when the whole is not explicitly identified in a given task. Some teachers need to have the one whole drawn out in order to reason about parts of that whole appropriately. For one task, we presented teachers the diagram shown below and asked for the areas of the large rectangle, one row, and each of the two shaded regions. The teachers we interviewed could calculate the correct product of $\frac{3}{4}$ x $1\frac{2}{5}$ and were familiar with using rectangular areas to model products of proper fractions, but they struggled with this task. For instance, several teachers thought that the area of the lightly shaded region was one whole. Other teachers had to extend the drawing to show two complete wholes in order to identify the areas of the shaded regions correctly. These difficulties indicate constrains on teachers' capacities to reason with parts of parts.

$\frac{3}{4}$



$1\frac{2}{5}$

Figure 3. An area model for $\frac{3}{4}$ x $1\frac{2}{5}$.

Comments/Boundaries
Teachers' difficulties with referent units in fraction multiplication situations may be related to their interpretations of fractions: Those who need to see a complete whole may be thinking of a fraction as a part-whole relationship rather than as a multiplicative relationship.

Attribute 1c: Referent Units for Division

Although most teachers can use algorithms to calculate the correct quotient of two fractions or two decimals, many studies have reported difficulties that U.S. teachers have with meanings for fraction division (e.g., Ball, 1990; Ma, 1999; Tirosh & Graeber, 1990). Tirosh and Gaeber reported that in one sample of 58 preservice elementary teachers, those that did have a meaning for fraction division had the partitive meaning. We have encountered teachers that had only the quotitive model (e.g., Izsák, Jacobson, de Araujo, & Orrill, in press). Although Ma reported cases of Chinese teachers who had both models for fraction division, the extent to which U.S. teachers have one or both models remains unclear. Simply having an appropriate meaning for division (quotitive or partitive) is not always sufficient to complete a division task when numbers are embedded in problem situations (e.g., Izsák et al., in press). Teachers also need to be able to understand the units to which numbers refer.

Example 1. Quotitive division
Quotitive division asks how many groups are formed when A objects (or units) are separated into groups of B objects (or units). One is measuring the group of A objects (or units) in terms of groups of B objects (or units). A and B can be whole numbers or fractions. Consider the following problem:

> John is making batches of cookies. Each batch requires $\frac{1}{4}$ cup of butter. He has $\frac{1}{3}$ of a cup of butter. How many batches can he make?

This question is asking how many $\frac{1}{4}$ cups are in $\frac{1}{3}$ cup. In the corresponding division statement, $\frac{1}{3} \div \frac{1}{4} = \frac{4}{3}$, the referent unit for $\frac{1}{3}$ is a standard unit for measurement, 1 cup. The referent unit for $\frac{1}{4}$ is a rate, $\frac{1}{4}$ cup of butter per batch. Strictly speaking the referent unit for $\frac{4}{3}$ is whole batches, but one can also think of the referent unit for $\frac{4}{3}$ as $\frac{1}{3}$ cups.

Example 2. Quotitive division
Izsák et al. (in press) reported a study of teacher learning in professional development during which a teacher used a repeated subtraction model for fraction division to explain quotients that were whole numbers. Using repeated subtraction, she could explain why $\frac{1}{3} \div \frac{1}{6} = 2$ with a drawn model. She knew that the answer, 2, referred to the number of $\frac{1}{6}$ there are in $\frac{1}{3}$. This same teacher had a hard time using drawn models to explain quotients that were not whole

numbers. For instance, she struggled to use rectangular areas to model the division statement $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$. She said that the problem asked how many times $\frac{3}{4}$ could be subtracted from $\frac{2}{3}$ but interpreted the quotient to mean $\frac{8}{9}$ of 1, rather than $\frac{8}{9}$ of $\frac{3}{4}$. Thus, the repeated subtraction model seems to support appropriate referent units only for some division problems.

Example 3. Partitive division

Partitive division asks how many objects (or units) are in each group when A objects (or units) are separated into B groups. Again, A and B can be whole numbers or fractions. Teachers are very familiar with the partitive model in case of whole numbers when it is interpreted as sharing, but there is little in the mathematics education research literature on U.S. teachers' understanding of the partitive model for fraction division. Consider the following problem:

> Susy is draining her bathtub. If it takes 1/3 of a minute to drain 1/4 of the bathtub, how many minutes does it take to drain the whole bathtub?

In the corresponding division statement, 1/3 ÷ 1/4 = 4/3, the referent unit for 1/3 is a standard unit for measurement, 1 minute. The referent unit for 1/4 is one bathtub, a unit for volume in this problem. The referent unit for 4/3 is a rate, minutes for one bathtub. In the DTMR interviews, some teachers recognized that they could use division to solve partitive division problems, but they could not explain why in terms of the presented problem situation. Other teachers recognized that problems like the bathtub problem can be solved using a proportion, but few of these teachers saw connections to fraction division. We see partitive division as a special case of proportional reasoning.

Comments/Boundaries

We include partitive division both because of the connection to proportional reasoning and because we hope that our measure will be used in conjunction with professional development that treats the partitive model for fraction division.

Attribute 2: Partitioning

The *Partitioning* attribute covers aspects of partitioning required for using drawn models to solve fraction arithmetic problems. Numerous researchers have highlighted the central importance of partitioning quantities in students' developing understanding of rational numbers (e.g., Confrey, 1994; Confrey & Smith, 1994, 1995; Empson, 1999; Empson, Junk, Dominguez, & Turner, 2005; Empson & Turner, 2006; Fosnot & Dolk, 2002; Hackenberg, 2007, 2010; Hackenberg & Tillema, 2009; Kieren, 1990; Lamon, 1996, 2007; Mack, 1993, 1995; Pitkethly & Hunting, 1996; Pothier & Sawada, 1983; Steffe, 2003, 2004; Streefland, 1991, 1993). Several researchers (e.g., Confrey, 1994; Confrey & Smith, 1994, 1995; Empson et al., 2005; Empson & Turner, 2006; Hackenberg & Tillema, 2009; Steffe, 2003, 2004) have examined relationships between students' partitioning and their conceptions of whole number multiplication. To the best of our knowledge, researchers have yet to examine closely teachers' capacities to partition. We use the only study of which we are aware (Mojica & Confrey, 2009) in an example below.

We are interested not only in partitions that divide a quantity into equal-sized pieces but also in nested levels of units in which units at one level in a partition have a fixed multiplicative relation with units at the next level. As an example, consider a 1-foot unit divided into four one-fourths, each of which contains three 12ths. Then three 12ths make up each fourth. In this example, the one whole is at the top level, but it does not have to be. As an example where the one whole is at the mid level, consider a 2-foot length where each foot is divided into four fourths, and each fourth is further divided into three parts. Then three 12ths make up each fourth, and four fourths make up each foot. As an example where the one whole is at the finest level, consider the base-10 system for whole numbers. The ones are nested in tens, tens are nested in hundreds, and so forth. We examine cases that do not make particular use of multiplicatively nested unit structures—simple partitioning—and cases that do make use of multiplicatively nested unit structures—partitioning in stages, partitioning using common denominators, and partitioning using common numerators.

Attribute 2a. Simple Partitioning

The term simple partitioning refers to partitioning with just two levels of units. The whole unit is one level divided into a certain number of equal-sized pieces which form the second level. We expect teachers to be adept at simple partitioning when the whole is given.

Example 1. Simple partitioning.
Into how many equal-sized parts would you need to cut 1 pizza so that Charles gets 3/5 as much pizza as Tony?

Here a teacher would need to see that 8 parts are needed and so the pizza should be divided into eighths.

Attribute 2b. Partitioning in Stages

The term *partitioning in stages* refers to creating an initial partition of some quantity and then a repartition. Whole number factor-product combinations can guide such partitioning activity. As an example, using the fact that 2 x 2 x 3 = 12, one could begin to divide a length into 12ths by dividing the length in half, then dividing the halves in two to create fourths, and finally dividing the fourths in three to create 12ths. Notice that partitioning in stages results in a three-level structure (12ths nested in fourths nested in the one whole). Past research (e.g., Hackenberg & Tillema, 2009; Steffe, 2003, 2004) has reported that forming three-level unit structures is difficult for students and plays a fundamental role in their capacities to use drawn models to reason about products of fractions. Less is known about teachers' capacities to form three-level unit structures. Izsák, Tillema, & Tunç-Pekkan (2008) reported a case in which forming three-level unit structures was difficult for in-service middle grades teachers and constrained their capacity to used drawn models of fraction multiplication with their students effectively. The teachers were using Connected Mathematics Project materials (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). Izsák et al. (in press) reported that some teachers in a professional development course appeared to reason with three levels of units, while others appeared to reason with just two levels of units.

Example 1. Partitioning in stages using whole-number factor-product combinations
In the only study of teachers' partitioning of which we are aware, Mojica and Confrey (2009) reported that even after instruction pre-service elementary teachers had trouble finding more than one method for folding a strip of paper into 12 equal-sized parts or into18 equal-sized parts. In our experiences, middle grades teachers are generally adept at breaking a given whole unit into a certain number of equal-sized parts, but they need connections between whole number multiplication and partitioning if they are to help students solve problems before instruction in standard algorithms for numeric computation.

Example 2. Partitioning in stages and multiplication of fractions
Partitioning in stages can be useful when using drawn quantities to reason about multiplication. For instance, *recursive partitioning* (Steffe, 2003, 2004) is defined to be taking a partition of a partition in the service of a non-partitioning goal. To understand the result of taking 1/4 of 1/3, students might begin by partitioning a unit into three pieces and continue by partitioning the first of those pieces into four smaller pieces (see Figure 4a). Determining the size of the resulting piece is a non-partitioning goal, and students could accomplish this in several ways. Students might see that concatenating 12 copies reconstructs the original unit (see Figure 4b). This solution requires constructing a unit of units (one unit containing 12 twelfths). This is a two-levels of units structure. Alternatively, students might recursively partition by subdividing each of the remaining thirds into four pieces (see Figure 4c). In contrast to the first solution, recursive partitioning involves constructing a unit of units of units (in the present example, one unit containing three thirds, each of which contains four twelfths). This is a three-levels of units structure because the thirds are maintained throughout. Other researchers have also reported that taking partitions of partitions has been central to students' construction of fraction concepts (e.g., Fosnot & Dolk, 2002; Kieren, 1990; Streefland, 1991, 1993).

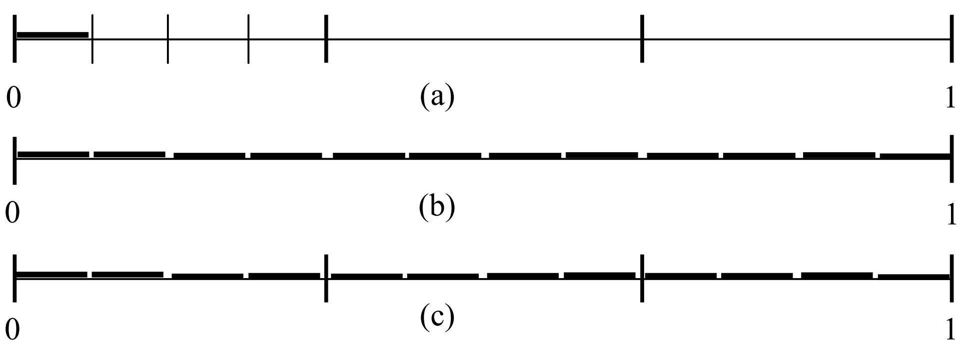


Figure 4. Determining 1/4 of 1/3. (a) Constructing part of a part. (b) Using two levels of units to determine 1/12. (c) Using three levels of units to determine 1/12.

Comments/Boundaries
In DTMR interviews have observed variation in teachers' capacities to coordinate multiple levels of units: Some teachers appear to coordinate three levels of units, while others appear to coordinate only two levels of units. We have been unable, however, to write items that reliably diagnose which teachers are coordinating three levels of units and which are only coordinating two levels of units. We note that the capacity to reason with three-level unit structures may make

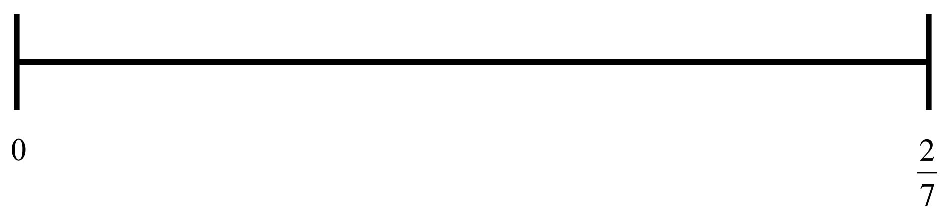
it easier to attend to appropriate referent units and to perform more complex partitioning. This could help explain strong correlation between the referent units and partitioning attributes, if we see such correlation in the national sample.

Attribute 2c. Partitioning Using Common Denominators

The term *partitioning using common denominators* refers to using knowledge of common denominators as a resource for guiding partitioning activity. Teachers know the importance of common denominators when using numerical methods for comparing, adding, or subtracting fractions. In DTMR interviews, however, we observed that teachers did not always recognize quickly that knowledge of common denominators could be used to partition drawn quantities in service of solving problems.

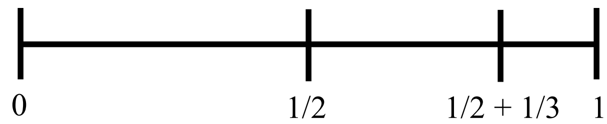
Example 1. Using common denominators to find common partitions
Teachers are generally adept at partitioning when the one whole is presented as part of the problem and they know the size pieces they need to construct. In cases where the one whole is not presented, teachers have to rely on other understandings to partition. The following task can be accomplished by focusing on a common partition of 7ths and 5ths:

> Subdivide the interval from 0 to 2/7 into equal-sized pieces to locate 1/5.

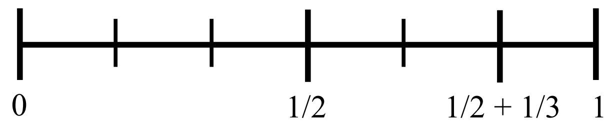


Using the common denominator of 1/5 and 2/7, teachers can partition the line segment into 10 parts (35thts), five of which comprise 1/7 and seven of which comprise 1/5. In this case, the levels of units are 35ths, 7ths, 5ths, and the (imagined) one whole. In DTMR interviews, some teachers saw this more quickly than others.

Example 2. Using common denominators to measure combined lengths
Consider the sum 1/2 + 1/3 (see Figure 5a). To measure a combined length of 1/2 and 1/3, one could create a structure of multiplicatively nested units that simultaneously subdivides halves and thirds of the whole. Because 6 is a common denominator, repartitioning the one whole into sixths will suffice. The partitioning in Figure 5b shows that 1/2 is the same length as 3/6, that 1/3 is the same length as 2/6, and that the sum is therefore 5/6. While teachers know the central role that common denominators play in comparing, adding, and subtracting fractions, students learning to add and subtract fractions oftentimes do not understand the need for common denominators. Finding common partitions like the one shown in Figure 5b can motivate the need for common denominators. Here the levels of units are 6ths, 3rds, halves, and the whole.



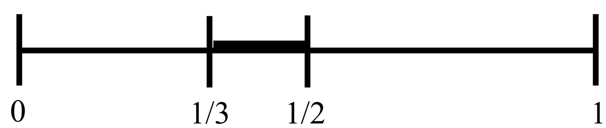
(a)



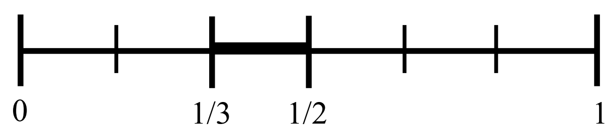
(b)

Figure 5. (a) A length of 1/2 + 1/3. (b) Nesting sixths into halves and thirds.

Example 3: Common denominators, referent units, and division of fractions
Knowledge of common denominators can aide partitioning quantities when solving fraction division problems. Consider the problem, "How many thirds are in one half?" To solve this problem, one can draw one third inside of one half (see Figure 6a). Answering the problem requires relating the bold segment to the length 1/3, which is an example of attending to referent units (Attribute 1c). Because 6 is a common denominator of 1/2 and 1/3—or a common multiple of 2 and 3—repartitioning the one whole into sixths will suffice. With this partition, one can see that the bold segment is 1/2 of 1/3, and therefore the answer is 3/2 (see Figure 6b). Here the levels of units are 6ths, 3rds, halves, and the whole.



(a)



(b)

Figure 6. (a) One third, inside of one half, inside of one whole. (b) Nesting sixths into halves and thirds.

Comments/Boundaries
In DTMR interviews we have seen teachers who knew that they could use common denominators to partition but had trouble doing so appropriately. As one example, a teacher

might know that the common denominator for $\frac{1}{4}$ and $\frac{1}{6}$ is 24 but then try to divide just the $\frac{1}{4}$ or the $\frac{1}{6}$ into 24 parts, suggesting difficulty with three-level unit structures.

Attribute 2d. Partitioning Using Common Numerators

The term *partitioning using common numerators* refers to using knowledge of common numerators as a resource for guiding partitioning activity. Unlike the notion of common denominators, which is explicitly discussed and emphasized in most fractions instruction, the notion of common numerators may be less familiar to teachers. Nevertheless, partitioning by common numerators can be useful for solving some problems, such as partitive division problems. Because we realize that many teachers will be less familiar with common numerators, we do not expect them to articulate this idea explicitly.

Example 1. Using common numerators in partitive division situations
artitioning using common numerators is useful when solving partitive division problems (and proportion problems more generally). Imagine a class in which students are solving a series of problems intended to culminate in learning the invert and multiply algorithm for fraction division. Students have solved several problems using fundraising thermometers when they turn to the following problem:

> Jamal's class is raising money for a school field trip. After 2/3 of the first month, the class has raised 5/7 of the money it needs. Assuming that the class is raising money at a steady pace, how many months will it take for the class to raise all the money it needs? Use a fundraising thermometer to answer the question.

Students might start by drawing the part of the thermometer that represents the amount of money raised thus far (Figure 7a). To continue the solution, they would need to partition the part of the thermometer in a way that simultaneously subdivides the thirds of months and the sevenths of money. This requires partitioning not by the common denominator, but by the common numerator, 10 (Figure 7b). Students could then iterate 1/7 of the money to determine that the money they need to raise corresponds with 14/15 of a month (Figure 7c).

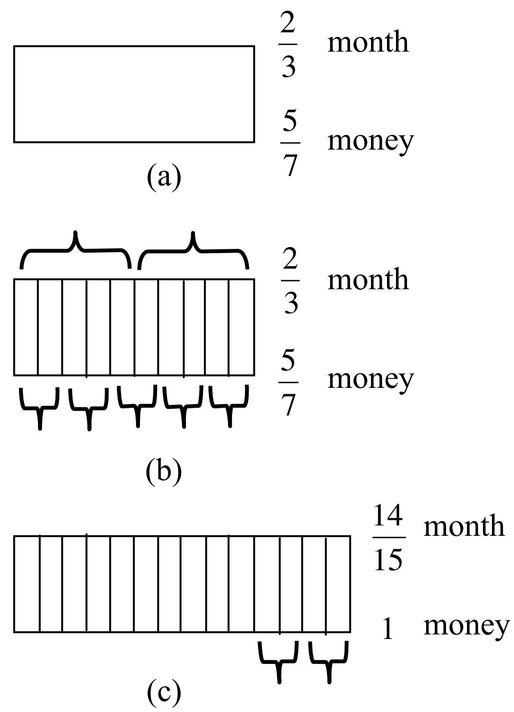


Figure 7. (a) A thermometer representation of 2/3 months and 5/7 of the money. (b) Subdividing thirds and sevenths. (c) Iterating to determined that the money will be raised in 14/15 months.

Comments/Boundaries

In the DTMR interviews, we observed that teachers could often partition using a single numerator. For instance, told that a given length is 4/3 units, many teachers can partition the length into 4 parts and take 3 of those parts to reconstruct the one whole. At the same time, we found that problems involving partitioning using common numerators were difficult for the teachers we interviewed. Because the notion of common numerators is unfamiliar to many teachers, we expect items that load onto this sub-attribute to be particularly difficult.

Attribute 3: Iterating

The *Iterating* attribute focuses on iterating unit fractions (fractions whose numerators are 1). Iterating can play an important role not only in solving problems but also in establishing meaning for fractions. Below we contrast two meanings for fractions.

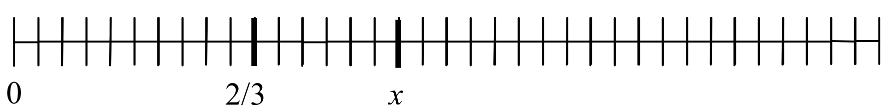
At least in the United States, the part-whole definition for fractions is used widely in school curricula. According to this definition $\frac{A}{B}$ is interpreted to mean a subset of cardinality A taken from a set of cardinality B. As an example, one might illustrate the meaning of $\frac{3}{4}$ by saying "three out of four cookies are chocolate chip." Several researchers (e.g., Ball, 1993; Izsák, 2008; Mack, 1990, 1993, 1995; Owens & Super, 1993; Streeflend, 1991) have reported students' use of this definition. A major limitation is that only proper fractions can be considered: It does not make sense for a subset to be larger than the original set. Nevertheless, in various projects, we have observed teachers use *A*-out-of-*B* language when working with students, when participating in professional development, and when participating in DTMR interviews.

Iterating unit fractions supports an alternative to the part-whole definition in which $\frac{A}{B}$ means A copies of the unit fraction one-Bth of some quantity M. Using this interpretation, one can interpret 3/8 as 3 one-eighths and 9/8 as 9 one-eighths. Thus, this definition does not create as much of an obstacle to understandings improper fractions, among other things. Although this second definition has not been used widely in the United States, Beckmann (2010) emphasized the A one-B^{ths} definition in *Mathematics for Elementary Teachers* (p. 39) as did the *Common Core State Standards* (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, see p. 24). Curricula used in Japan introduce fractions using the A one-B^{ths} definition (Hironaka & Sugiyama, 2006, see book 3B). Curricula used in Singapore introduce fractions using the part-whole definition but switch to the A one-B^{ths} definition for improper fractions (Curriculum Planning & Development Division, Ministry of Education Singapore, 2003, see books 3B and 4A). Thompson and Saldanha (2003) argued the benefits of the A one-B^{ths} of M interpretation, and past research (Steffe, 1993, 2001; Tzur, 1999, 2000, 2004) has demonstrated the constructive role that iterating unit fractions can play in students' construction of fraction knowledge. Teachers need to be adept at iterating unit fractions so that they can facilitate this fundamental way of thinking in their students.

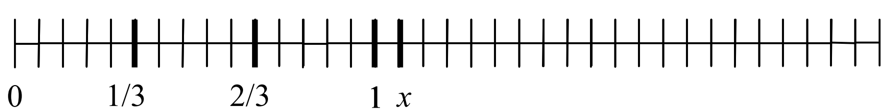
Example 1. Iterating to establish the whole and improper fractions

The following example is derived a task discussed by Lamon (2005, p. 175) and contains two applications of iterating a unit fraction: one application to determine the location of the one whole and a second application to determine an improper fraction.

Given the point 2/3, determine point *x*.



First, if we think of 2/3 as 2 (1/3 length units), we can partition the 2/3 length in half to determine the 1/3 unit and then iterate the 1/3 unit three times to find the one whole length. This interpretation emphasizes the relative size of 2/3 to 1.



Second, to determine the point *x*, we could see that the one whole is partitioned into 15 parts and that iterating one of those parts 16 times establishes that *x* is 16/15. This interpretation emphasizes the relative size of 16/15 to 1.

Example 2. Using common denominators to find a common unit fraction, iterating, and attending to referent units

Finding common unit fractions is a useful strategy in many contexts. As one example, consider once more the following division problem:

> John is making batches of cookies. Each batch requires $\frac{1}{4}$ cup of butter. He has $\frac{1}{3}$ of a cup of butter. How many batches can he make?

One way to draw a model for this problem is shown below:

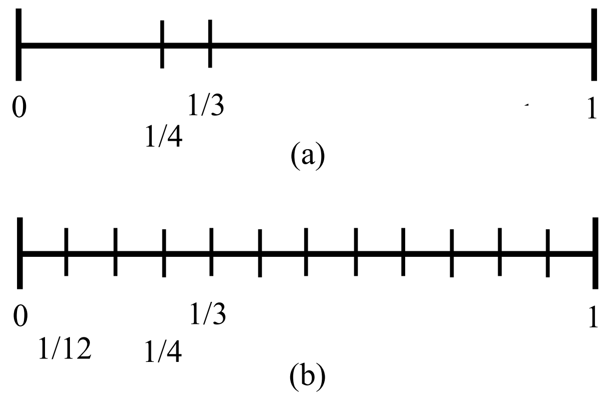


Figure 8. (a) A linear model showing $\frac{1}{4}$ and $\frac{1}{3}$. (b) $\frac{1}{12}$ as a common unit fraction for $\frac{1}{4}$ and $\frac{1}{3}$.

Because $\frac{1}{4}$ does not divide $\frac{1}{3}$ evenly, a common unit fraction is needed in order to understand the multiplicative relationship between these two fractions. Common denominators can be used to partition in terms of common unit fractions. In the example above, because 4 and 3 are factors of 12, $\frac{1}{12}$ can be iterated 3 times to exhaust $\frac{1}{4}$ and 4 times to exhaust $\frac{1}{3}$. This implies that the answer is $\frac{4}{3}$. A complete solution to the problem also requires attention to referent units: The answer is $\frac{4}{3}$ of $\frac{1}{3}$.

Example 3. Partitioning and iterating in the context of proportional reasoning

Combining partitioning with iterating is important for reasoning about both fractions and proportions. Consider the following proportion problem:

> One batch of a certain shade of purple paint is made by mixing 3 pails of blue paint with 2 pails of red paint. If I have 5 pails of blue paint, how many pails of red paint do I need to make the same shade of purple? (InterMath, http://intermath.coe.uga.edu)

We watched a teacher in a professional development course solve this problem using a double number line. He used one number line for blue paint and one for red paint (see Figure 9a). The tick marks for 0 on both number lines are aligned, as are the tick marks for 3 pails of blue paint and 2 pails of red paint, for 6 pails of blue paint and 4 pails of red paint, and for 9 pails of blue paint and 6 pails of red paint. He explained that he used the common multiple 6 to partition one batch made with 3 pails of blue paint and 2 pails of red paint. His use of whole number multiplication to guide his partitioning is similar to partitioning in stages, using common denominators, and common numerators (Attributes 2b, 2c, and 2d). In this case, he partitioned the 3:2 unit to create the 0.5:0.33 unit. He then iterated the 0.5:0.33 unit to create the following sequence of composed units: 3.5:2.33, 4:2.66, 4.5:3, 5:3.33, and finally 5.5:3.66 (see Figure 9b). He concluded that 5 pails of blue paint require 3.33 (more precisely 3 1/3) pails of red paint.

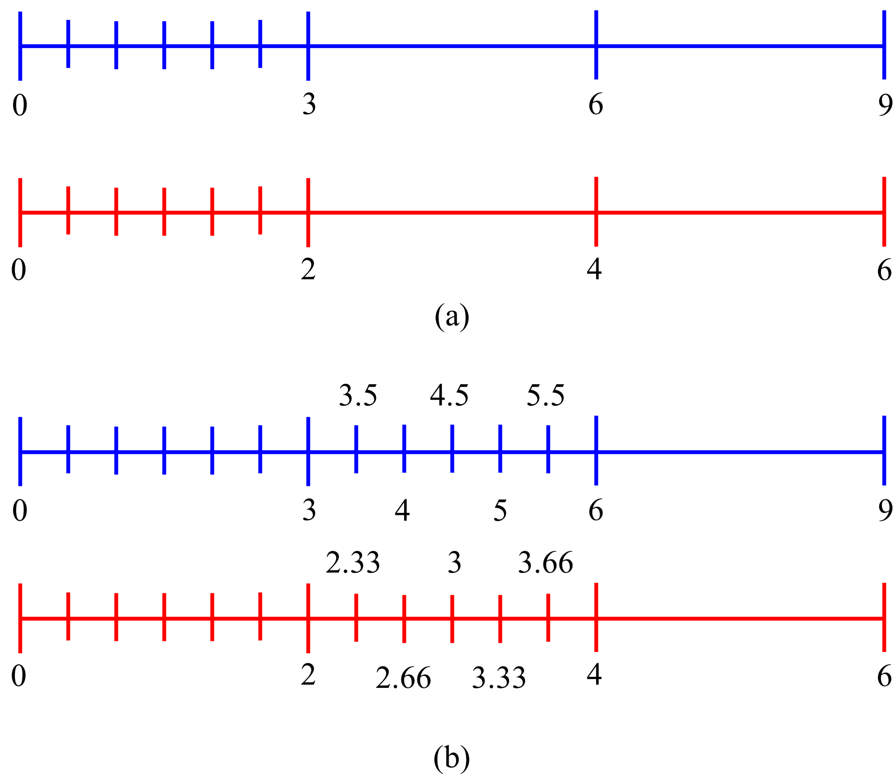


Figure 9. (a) Using multiplication to partition a composed unit. (b) Combining the 3:2 unit with four copies of the 0.5:0.33 unit to create the 5:3.33 unit.

Comments/Boundaries

None.

Attribute 4: Appropriateness

The *Appropriateness* attribute refers to identifying an appropriate operation or mathematical expression for a given problem situation. Doing so requires identifying a relationship among quantities in the situation (perhaps presented through a word problem or a diagram) and associating this quantitative relationship with an appropriate arithmetical operation. Past research has demonstrated that preservice and inservice elementary teachers (including Grade 6 teachers) can have difficulty identifying the appropriate operation in multiplication and division situations (e.g., Graeber & Tirsoh, 1988; Harel & Behr, 1995; Harel, Behr, Post, & Lesh, 1994; Tirsoh & Graeber, 1990). Much of this work has been framed by work on intuitive models for arithmetic operations (Fischbein, Deri, Nello, & Marino, 1985) and has focused on decimals. Teachers should be able to identify multiplication and division situations that use a range of number types (e.g., whole numbers, fractions, and decimals), various combinations of number sizes (e.g., a smaller number divided by a larger number and vice versa), and in a range of contextual settings (e.g., continuous as well as discrete settings). We examine three cases of appropriateness: identifying multiplication situations, identifying quotitive division situations, and identifying partitive division situations. The proportional reasoning attributes also include an appropriateness attribute that covers identifying situations that are direct proportions.

Attribute 4a. Identifying Multiplication Situations

Identifying multiplicative situations requires recognizing a quantitative structure either involving A groups of size B or a multiplicative "times as many" comparison. A and B can be whole numbers, fractions, or decimals. Teachers are generally able to identify multiplicative situations when all the numbers are whole numbers. Working with whole number multiplication can lead to two intuitive rules: (a) The multiplier must be a whole number and (b) The product must be larger than either of the two numbers being multiplied together. Past research has demonstrated that teachers can struggle to discriminate between multiplication and division situations, especially when multiplying by decimals between 0 and 1 (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Harel & Behr, 1995; Post, Harel, Behr, & Lesh, 1991; Tirosh & Graeber, 1989). In case of fractions, we have seen in DTMR interviews that teachers often rely on an association between the word "of" and multiplication to identify multiplication situations.

Example 1:
Graeber and Tirosh (1988) reported that in a sample of 129 preservice elementary teachers most could write appropriate expressions that conformed to the two intuitive rules discussed above, but only 59% answered the problem below correctly. The most common error was to use division inappropriately.

> One kilogram of detergent is used in making 15 kilograms of soap. How much soap can be made from .75 kilograms of detergent? (Graeber & Tirosh, 1988, p. 264)

Example 2:
Harel et al. (1994, p. 376) reported that less than 60% of 293 preservice and 167 in-service elementary teachers solved word problems involving products like 0.75 x 0.62 correctly. The researchers did not provide the exact word problems they used, but an example word problem follows.

> A sea turtle swims 0.62 kilometers in ten minutes. A second sea turtle swims 0.75 the distance that the first turtle swam. How far did the second turtle swim?

Attribute 4b. Identifying Quotitive Division Situations

Identifying quotitive division situations requires recognizing a quantitative structure in which one quantity is measured in terms of the other. A typical question is how many groups are formed when A objects (or units) are separated into groups of B objects (or units). In this case, one is measuring the group of A objects (or units) in terms of groups of B objects (or units). Again, A and B can be whole numbers, fractions, or decimals. Teachers are often able to identify quotitive division situations that involve whole numbers. The quotitive model with whole numbers can lead to the intuitive rule that the divisor should be smaller than the dividend (e.g., Harel et al., 1994, p. 365). Past research has demonstrated that teachers have trouble constructing fraction division situations (e.g., Ball, 1990; Borko et al., 1992; Ma, 1999).

Example 1:
Harel et al. (1994, p. 376) reported that word problems violating the intuitive rule that the divisor should be smaller than the dividend were much more difficult for preservice and inservice teachers than problems that did not violate this rule. The researchers did not provide the exact word problems they used, but an example word problem follows.

> Roger is driving 320.6 miles to visit his brother. He has already driven 95.3 miles. How much of the trip has he completed?

Attribute 4c. Identifying Partitive Division Situations

Identifying partitive division situations requires recognizing a quantitative structure in which the quotient, a certain amount of one quantity, is associated with one unit of a second quantity. A typical question is how many objects (or units) are in each group when A objects (or units) are separated into B groups. Again, A and B can be whole numbers, fractions, or decimals. Teachers are usually very familiar with fair sharing that is partitive division with whole numbers, but they may not be aware that partitive division affords a special case of proportional reasoning. The partitive model with whole numbers can lead to three intuitive rules: (a) The divisor should be a whole number, (b) The divisor should be smaller than the dividend, and (c) Division makes the quotient smaller than the dividend (e.g, Harel et al., 1994, p. 365). Ma (1999) reported Chinese teachers' use of the partitive model for fraction division, but both in her report and our own experience few U.S. teachers are familiar with these situations or the connection between fraction division and proportional reasoning.

Example 1:
Graeber and Tirosh (1988) reported that in a sample of 129 preservice elementary teachers most could write appropriate expressions for problems that conformed to the three intuitive rules discussed above, but only 34% generated an appropriate expressions for the problem below correctly. The most common error was to invert the dividend and the divisor (e.g., they wrote 12 ÷ 5).

> Twelve friends together bought 5 pounds of cookies. How many pounds did each get if they each got the same amount (Graeber & Tirosh, 1988, p. 265)

Example 2:
Harel et al. (1994, p. 376) reported that word problems violating the first intuitive rule—that the divisor should be a whole number—were much more difficult for preservice and in-service teachers than problems that violated the second intuitive rule—that the divisor should be greater than the dividend. The researchers did not provide the exact word problems they used, but an example word problem follows.

> Jason is buying biscuits for his dog at the pet store. Biscuits are sold by the pound. If 6 biscuits weigh 0.67 pounds. How many biscuits will weigh 1 pound?

Comments/Boundaries:
The common practice of using "keywords" to select operations allows students and teachers to solve problems without examining relationships among quantities closely.

Attribute 5: Fractions as Multiplicative Comparisons

In general, a multiplicative comparison is formed by asking "How many times as great is one value than another?" or "What portion or fraction of one value is another?" (Thompson, 1994). These comparisons differ from additive comparisons which are formed by asking "How much more is one thing than another" or "How much less is one thing than another?" For example, suppose the heights of two boxes are 9 in. and 6 in. We can compare the heights *multiplicatively* by saying that one box is 1 ½ times the height of the other or that one box is 2/3 the height of the other. Alternatively, we can compare the heights *additively* by saying that one box is 3 in. taller than the other or that one box is 3 in. shorter than the other. Multiplicative comparisons can be formed using any real number—including whole numbers, proper fractions (including unit fractions), and improper fractions. It can be difficult to infer when a teacher has formed a multiplicative comparison. One indicator that a teacher has formed such a comparison is if he or she explicitly compares two quantities multiplicatively—for instance, by using phrases like "times as many." One indicator that a teacher has not formed a multiplicative comparison is when he or she cannot make sense of a question that involves a multiplicative comparison: We have observed teachers have difficulty interpreting expressions such as "9/5 of the amount."

Example 1:

> A cyclist rides 75 miles in one day. Her friend drives 135 miles to meet her. How many times further did the driver travel than the cyclist?

Answering this question correctly requires being able to conceive of the fraction 9/5 as describing a multiplicative comparison: The driver travels 9/5 times as much as the cyclist travels or 9/5 the distance that the cyclist travels. In our experience, teachers are more comfortable using whole numbers to make multiplicative comparisons than they are using fractions to make multiplicative comparisons.

Comments/Boundaries:
Using the key word "of" to recognize when to multiply by a fraction is not the same as forming a multiplicative comparison. Multiplicative comparisons are also important in proportional reasoning and show up as sub-attributes for the proportional reasoning test as well.

References

Armstrong, B., & Bezuk, N. (1995). Multiplication and division of fractions: The search for meaning. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 85-119). Albany, NY: State University of New York Press.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144.

Ball, D. L. (1993). Halves pieces and twoths: Constructing representational contexts in teaching fractions. In T. P. Carpenter (Ed.), *Rational numbers* (pp. 157-195). New York: Erlbaum.

Beckmann, S. (2010). *Mathematics for elementary teachers with activity manual* (3rd ed.). Boston: Addison-Wesley.

Behr, M., Harel, G., Post, T., & Lesh, R. (1992). Rational number, ratio, and proportion. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 296-333). New York: Macmillan.

Behr, M. J., Wachsmuth, I., Post, T. R, & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.

Confrey, J. (1994). Splitting, similarity, and rate of change: A new appraoch to multiplication and exponential functions. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 291-330). Albany: State Univeristy of New York Press.

Confrey, J., & Smith, E. (1994). Exponential functions, rates of change, and the multiplicative unit. *Educational Studies in Mathematics, 26*(2), 135-164.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Curriculum Planning and Development Division, Ministry of Education Singapore. (2003). *Primary mathematics*. Singapore: Times Media Private Ltd. (Available at *www.singaporemath.com*)

Eisenhart, M., Borko, H., Underhill, R., Brown, C., Jones, D., & Agard, P. (1993). Conceptual knowledge falls through the cracks: Complexities of learning to teach mathematics for understanding. *Journal for Research in Mathematics Education*, 24(1), 8-40.

Empson, S. (1999). Equal sharing and shared meaning: The development of fraction concepts in a first-grade classroom. *Cognition and Instruction*, 17(3), 283–342.

Empson, S., Junk, D., Dominguez, H., & Turner, E. (2005). Fractions as the coordination of multiplicatively related quantities: A cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*(1), 1–28.

Empson, S., & Turner, E. (2006). The emergence of multiplicative thinking in children's solutions to paper folding tasks. *Journal of Mathematical Behavior, 25*(1), 46–56.

Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*(1), 3-17.

Fosnot, C. T., & Dolk, M. (2002). Young mathematicians at work: Constructing fractions, decimals, and percents. Portsmouth, NH: Heinemann.

Graeber, A., & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary

teachers' performance and beliefs. *Journal of Mathematical Behavior, 7*, 263-280.
Graeber, A., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 20*(1), 95-102.
Hackenberg, A. J. (2007). Units coordination and the construction of improper fractions: A revision of the splitting hypothesis. *Journal of Mathematical Behavior, 26*(1), 27–47.
Hackenberg, A. J. (2010). Students' reasoning with reversible multiplicative relationships. *Cognition and Instruction, 28*(4), 383–432.
Hackenberg, A. J., & Tillema, E. S. (2009). Students' whole number multiplicative concepts: A critical constructive resource for fraction composition schemes. *Journal of Mathematical Behavior, 28*, 1–18.
Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education, 3*, 31-51.
Harel, G., Behr, M., Post, T., & Lesh, R. (1994). The impact of the number type on the solution of multiplication and division problems: Further considerations. In G. Harel & J. Confrey (Eds.), *The development of multiplcative reasoning in the learning of mathematics* (pp. 363–384). Albany, NY: State University of New York Press.
Hironaka, H., Y& Sugiyama. Y. (Eds.). (2006). *Mathematics for elementary school*. Tokyo Shoseki Co., Ltd. (Available at *www.golobaledresources.com*)
Izsák, A. (2008). Mathematical knowledge for teaching fraction multiplication. *Cognition and Instruction*, 26, 95-143.
Izsák, A., Jacobson, E., de Araujo, Z., & Orrill, C.. H. (in press) *Measuring growth in mathematical knowledge for teaching fractions with drawn quantities*. Journal for Research in Mathematics Education.
Izsák, A., Tillema, E., & Tunç-Pekkan, Z. (2008). Teaching and learning fraction addition on number lines. Journal for Research in Mathematics Education, 39(1), 33–62.
Kieren, T. E. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 53-92). Reston, VA: National Council of Teachers of Mathematics.
Kieren, T. (1990). Children's mathematics/Mathematics for children. In L. P. Steffe & T. Wood (Eds.), *Transforming children's mathematics education: International perspectives* (pp. 323–333). Hillsdale, NJ: Lawrence Erlbaum Associates.
Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 49-84). Hillsdale, NJ: Lawrence Erlbaum Associates.
Lamon, S.J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany, NY: State University of New York Press.
Lamon, S. (1996). The development of unitizing: Its role in children's partitioning strategies. *Journal for Research in Mathematics Education, 27*(2), 170–193.
Lamon, S. J. (2005). *Teaching Fractions and Ratios for Understanding* (2nd Ed.). Mahwah, NJ: Lawrence Erlbaum Associates.
Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 629–667). Charlotte, NC: Information Age Publishing.

Lappan, G., Fey, J. T., Fitzgerald, W., Friel, S. N., & Phillips, E. D. (2002). *Connected mathematics*. Glenview, IL: Prentice Hall.

Lappan, G., Fey, J. T., Fitzgerald, W., Friel, S. N., & Phillips, E. D. (2006). *Connected mathematics 2*. Boston, MA: Prentice Hall.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates.

Mack, N. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16–32.

Mack, N. (1993). Learning rational numbers with understanding: The case of informal knowledge. In T. Carpenter, E. Fennema & T. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 85–105). Hillsdale, NJ: Erlbaum.

Mack, N. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, *26*, 422–441.

Mojica, G. F. & Confrey, J. (2009). Pre-service elementary teachers' understanding of an equipartitioning learning trajectory. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, pp. 129–136. Atlanta, GA: Georgia State University.

National Governors Association Center for Best Practices and the Council of Chief State School Officers. (2010). Common core state standards for mathematics. Retrieved May, 2011, from http://www.corestandards.org/the-standards/mathematics.

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses* (pp. 1-54). Charlotte, NC: Information Age Publishing, Inc. and the National Council of Teachers of Mathematics.

Owens, D., & Super, D. (1993). Teaching and learning decimal fractions. In D. Owens (Ed.), *Research ideas for the classroom: Middle grade mathematics*. New York, NY: Macmillan Publishing Co.

Pitkethly, A., & Hunting, R. (1996). A review of recent research in the area of initial fraction concepts. *Educational Studies in Mathematics, 30*(1), 5–38.

Post, T., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rationale number concepts. In E. Fennema, T. Carpenter & S. Lamon (Eds.), *Integrating research on teaching and learning mathematics* (pp. 177-198). Albany, NY: State University of New York Press.

Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education, 14*(5), 307-317.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Lawrence Erlbaum.

Sowder, J., Philipp, R., Armstrong, B., & Schappelle, B. (1998). *Middle-grade teachers' mathematical knowledge and its relationship to instruction: A research monograph*. Albany: State University of New York Press.

Steffe, L. P. (1988). Children's construction of number sequences and multiplying schemes. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 119-140). Reston, VA: Lawrence Erlbaum Associates & National Council of Teachers of Mathematics.

Steffe, L. (1993, April). Children’s construction of iterative fraction schemes. Paper presented at the annual meeting of the national Council of teachers of Mathematics, Seattle, WA.

Steffe, L. (2001). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 20*(3), 267-307.

Steffe, L. (2003). Fractional commensurate, composition, and adding schemes learning trajectories of Jason and Laura: Grade 5. *Journal of Mathematical Behavior*, 22, 237–295.

Steffe, L. (2004). On the construction of learning trajectories of children: The case of commensurate fractions. *Mathematical Thinking and Learning*, 6, 129–162.

Streefland, L. (1991). *Fractions in realistic mathematics education: A paradigm of developmental research*. Dordrecht, The Netherlands: Kluwer Academic.

Streefland, L. (1993). Fractions: A realistic approach. In T. Carpenter, E. Fennema, & T. Romberg (Eds.), Rational numbers: An integration of research (pp. 289–325). Hillsdale, NJ: Lawrence Erlbaum Associates.

Thompson, P.W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.

Thompson, P. W., & Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J. Kilpatrick, W. G. Martin, & D. Shifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

Tirosh, D., & Graeber, A. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*, 79-96.

Tirosh, D., & Graeber, A. (1990). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education, 21*(2), 98–108.

Tzur, R. (1999). An integrated study of children's construction of improper fractions and the teacher's role in promoting the learning. *Journal for Research in Mathematics Education, 30*(4), 390-416.

Tzur, R. (2000). An integrated research on children's construction of meaningful, symbolic, partitioning-related conceptions, and the teacher's role in fostering that learning. *Journal of Mathematical Behavior, 18*(2), 123–147.

Tzur, R. (2004). Teacher and students' joint production of a reversible fraction conception. *Journal of Mathematical Behavior, 23*(1), 93–114.

DTMR Attributes for Proportional Reasoning

Joanne Lobato, Chandra Orrill, & Erik Jacobson

Background: Teacher's Knowledge of Proportional Reasoning

As a field, our knowledge of teachers' proportional reasoning is much less detailed than our knowledge of students' proportional reasoning. Research on teachers' knowledge of ratios and proportions has primarily (a) reported cases in which teachers have struggled to understand concepts related to proportionality (e.g., Cramer, Post, & Currier, 1993); (b) bootstrapped models of students' proportional reasoning to make sense of teachers' understanding (e.g., Hull, 2000); or demonstrated increased subject matter knowledge as a result of professional development (Ben-Chaim, Keret, & Ilany, 2007). At the outset of this project, we assumed that the extensive knowledge base on student capacities and misconceptions related to proportionality could form the foundation from which we constructed attributes and test items for teachers. However, after writing an initial set of items and conducting item response interviews with teachers, we realized that we needed to investigate the resources teachers have for proportional reasoning that are not typically accessible to students. We also wanted to better understand teachers' capacities to build on students' thinking and to enable the learning of essential mathematics by their students.

Consequently, we conducted clinical interviews (Ginsburg, 1997) with 14 middle grades teachers from four school districts in Georgia in November 2009. The protocol consisted of open-ended complex problems set in ecologically valid contexts, such as responding to: (a) student reasoning, (b) a teaching situation, or (c) a question from a fellow teacher. We have relied to a large extent on the resulting data to craft attributes and items for assessing teachers' proportional reasoning.

This document elaborates four attributes for proportional reasoning that form the foundation for the items on the DTMR proportional reasoning assessment form. Table 1 summarizes these

Attribute	Sub-attributes
Covariation & Invariance	Iterating and partitioning a composed unit
	Consolidating operations on composed units
	Making multiplicative comparison within measure spaces
	Making multiplicative comparison across measure spaces
Connections between Ratios & Fractions	Using composed unit reasoning to reinterpret a ratio as a fraction
	Using a multiplicative comparison to reinterpret a ratio as a fraction
	Differentiating fraction and ratio operations
Appropriateness	(no subcategories)
Ratios in Context as a Network of Related Quantities	Forming a ratio-as-measure
	Interpreting the meaning of equality of ratios
	Making connection with linear functions

Table 1. Proportional Reasoning Attributes and Sub-Attributes

attributes and the sub-attributes that contribute to each attribute. Because three of the attributes are at a macro-level in terms of grain-size, we found it useful to identify subcategories corresponding to more micro-level understandings for those attributes. Additionally, we use the sub-attributes as a tool for insuring that we assess each attribute in a variety of contexts. For each attribute and sub-attribute, we provide a general description of the understanding, an example of reasoning that exhibits the understanding, and an elaboration of the variety of ways that limited understanding of the attribute can instantiate itself.

While this document focuses on attributes related to ratios and proportions, woven throughout is an emphasis on *quantitative reasoning* (Smith & Thompson, 2008; Thompson, 1994). A quantity is one's conception of measurable features of objects, events, or situations (e.g., conceiving of distance, weight, how fast something travels, and so on). Quantitative reasoning involves analyzing the quantities and relationships among quantities in a situation, creating new quantities, and making inferences with quantities. Most of the attributes below depend on the presence of quantitative reasoning.

Proportional Reasoning Attributes

Attribute 1: Covariation and Invariance

According to Lamon (2007), proportional relationships involve the transformation of quantities in such a way that the mathematical structure is invariant. Middle grades teachers should understand that when two quantities, w and z, are related proportionally, then the following two invariant relationships hold:

- There is a constant of proportionality, k, by which $wk = z$ for all corresponding values of w and z.
- If w is increased or decreased by a factor of a/b, then z must increase or decrease by the same factor to maintain the proportional relationship.

There is some correspondence between these two relationships and what Vergnaud (1983, 1988) calls "across-measure space reasoning" and "within-measure space reasoning," respectively. A measure space can be thought of as the range of values that a particular measureable aspect of a situation can take on. For example, distance and time can be viewed as two measure spaces in a constant speed situation. Learners may focus within measure spaces (e.g., by forming ratios of distance to distance and time to time) or coordinate quantities across measure spaces (e.g., by forming a ratio of distance to time). While some researchers consider across-measure space strategies to be more sophisticated, according to a review of the everyday math literature on proportional reasoning, within-measure space reasoning is used much more frequently than across-measure space reasoning by practitioners working in everyday situations (Hoyles, Noss, & Pozzi, 2001). Thus, both types of reasoning are important.

Although the across-measure space and within-measure space distinctions are useful, we focus on a smaller grain-size of understanding in this attribute document, specifically on two ways to form a ratio from the perspective of quantitative reasoning—the formation of *composed units* and *multiplicative comparsions.* Teachers may form a ratio by joining or composing two quantities to create a new unit, which in turn, can be operated upon by iterating, partitioning, or splitting (Lamon, 1994, 1995). Alternatively, teachers may form a ratio as a multiplicative comparison of two quantities, by making a relative comparison of how many times as great one quantity is than another (Kaput & Maxwell-West, 1994; Thompson, 1994). While operations on composed units emphasize within-measure space reasoning, one can form multiplicative comparisons within or across measure spaces. Additionally, there are within- and across-measure spaces strategies that lack the quantitative reasoning component of either composed units or multiplicative comparisons, as is elaborated in sub-attributes 1A-1D below.

If we focused on middle school students' understandings (rather than on their teachers' understandings), then we would make more prominent the distinction between non-ratio and ratio reasoning. Research indicates that it is common for middle grades students to attend to a single quantity when reasoning in a proportional situation, which is known as *univariate* reasoning (Harel, Behr, Lesh, & Post, 1994; Piaget, 1952). Other students may coordinate the quantities but fail to preserve the multiplicative relationship between them by making absolute rather than relative comparisons; hence reasoning *additively* (Noelting, 1980). Although there were a few instances of additive reasoning in our clinical interviews, the teachers brought more conceptual resources to bear on the situations than their students. Of greater issue were the nature, extent, and flexibility of the components of understanding of covariation and invariance elaborated in the four sub-attributes below.

Attribute 1A: Iterating and Partitioning a Composed Unit

The teacher with this understanding constructs a rudimentary form of a ratio by joining together two quantities into a single entity, called a *composed unit*, which, in turn, can be operated upon. A teacher can preserve the invariance of ratio by *iterating* (joining together replicates of a quantity to produce a partitioned whole) and *partitioning* the composed unit to find other equivalent ratios (separating a quantity into a specified number of equal parts while the quantity remains as a whole) (Lobato & Ellis, 2010). Specifically, the teacher can find any whole-number iterates (e.g., tripling, finding 10 groups of, etc), partial groups (e.g., 1/4 of the composed unit), and any number of partitions (e.g., partitioning into 5 equal groups). According to Lamon (1994), one's "ability to think about a ratio as an invariant composite unit and work simultaneously with both its composite units in a double-matching process (covariance) illustrated the kind of understanding we would like them to have about the meaning of ratio and proportion" (1994, p. 112).

It can be difficult to infer when a teacher has formed a composed unit as opposed to operating on each quantity separately. This is because operating on a composed unit (e.g., by tripling it) is accomplished by operating on each of the constituent quantities (e.g., tripling each quantity). Indicators of joining two quantities mentally include (a) using some term such as "batch" to suggest an entity comprising two quantities and talking about operating on the "batch"; (b) regular juxtaposition of two quantities in written reports or in diagrams; (c) "coupling" gestures; and (d) verbally pairing the quantities.

Example. Consider the following problem derived from a nursing study by Holyes, Noss, and Pozzi (2001):

> *A drug comes in packets of 120 mg diluted in 2 ml of fluid. How much diluted drug should be administered for a dose of 300 mg?*

A teacher may solve this problem by first joining 120 mg and 2 ml into a composed unit, which we denote as 120:2. She then partitions the unit in half to obtain a new 60:1 unit. The 60:1 unit is iterated five times to arrive at 300:5, and the answer is 5 ml.

Limited Understandings. In our clinical interviews with teachers, we found instances of the following types of limited understandings related to this sub-attribute:

- Teachers appeared to form a composed unit but only use whole number iterates.
- Some teachers appeared to have formed a composed unit but only performed simple partitions, such as taking halves, and struggled with more difficult partitions, such as taking thirds.
- A few teachers doubled or halved each quantity separately and did not provide verbal, written, or gestural evidence of joining the two quantities into a unit.
- Finally, some teachers engaged in both additive reasoning and simple composed-unit reasoning (e.g., doubling and halving a composed unit) while working on a single task.

Attribute 1B: Consolidating operations on composed units

Sub-attribute 1B addresses the ratio understanding exhibited when a teacher combines iterates and partitions on a composed unit into a single factor *a/b* and uses multiplication or division to express the factor. The formation of a composed unit (sub-attribute 1A above) is a foundational concept that is not, by itself, indicative of sophisticated ratio reasoning. In fact, some researchers have referred to the formation of a composed unit as *pre-ratio* reasoning (Lesh, Post, & Behr, 1988). However, Lobato and Ellis (2010) argue that composed unit reasoning can support more sophisticated proportional reasoning by reflecting upon the number of groups that are created when iterating and partitioning and by combining quantitative operations, as indicated in the example below.

Example. Consider the following task from Lobato and Ellis (2010):

> *Begin with a ramp that is 3 cm high and has a base that is 4 cm long. What is the height of a new ramp with a base of 5 cm and the same steepness as the original ramp?*

We describe two different approaches, one from Teacher A (who we infer has an understanding of sub-attribute 1A) and one from Teacher B (who we infer has an understanding of sub-attribute 1B). Teacher A realizes that the base of the new ramp is 1 cm more than the base of the original ramp. She decides to find the height of a ramp with a base of 1 cm and the same steepness as the original ramp. She partitions the 3 : 4 original ramp into 4 equal parts to obtain a 3/4 : 1 ramp (see Figure 1). She then iterates and stacks the 3/4 : 1 ramp five times so that the base of the new ramp is 5 cm. The height new ramp is 15/4, or 3 ¾ cm, since it contains five ramps, each with a height of 3/4 cm (see Figure 2).

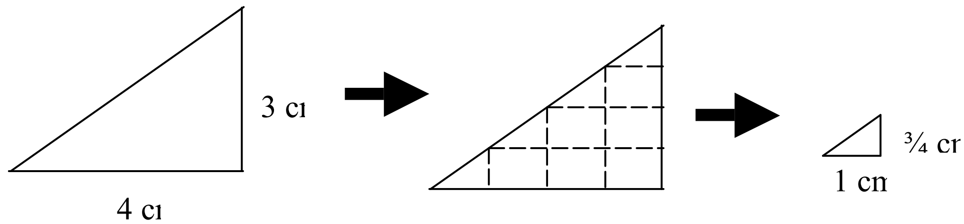


Figure 1. Partitioning a 3:4 unit into 4 equal parts

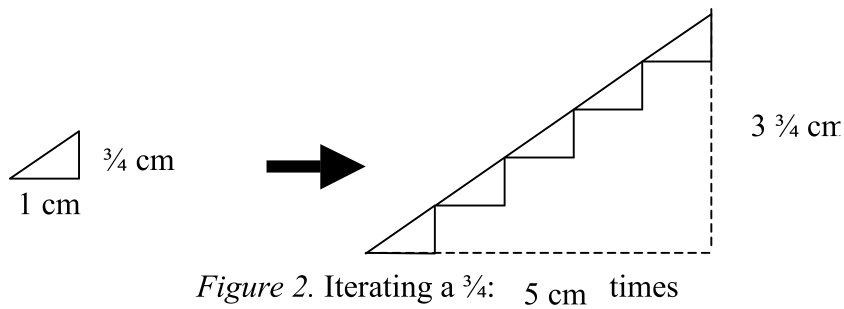


Figure 2. Iterating a ¾: times

Teacher B consolidates all of the operations used by Teacher A into a single operation. She realizes that 5 cm (the base of the new ramp) is 5/4 of 4 cm (the base of the original ramp). This teacher can identify the factor 5/4 by reflecting on her use of iterating and partitioning (e.g., 5/4 x 4 is the consolidation of 5 groups of ¼ of 4). Once the teacher realizes that 5 cm is 5/4 x 4 cm, she can complete the problem by finding 5/4 x 3, which is 15/4 or 3 ¾ cm (see Figure 2). Teacher B's reasoning provides support for the idea that one can maintain a proportional relationship by multiplying each quantity by the same factor *a/b*.

Limited Understandings. This sub-attribute was surprisingly challenging for the participating teachers in our interviews. Some teachers could consolidate operations but only for whole number iterates. Specifically a teacher would know that doubling and then tripling a composed unit is the same as multiplying that unit by a factor of six but would have difficulty working with fractional factors. Other teachers could locate the correct factor, as the result of performing an arithmetic operation such as division, but could not link the factor to the consolidation of a composed unit. Furthermore, some teachers were not sure that combining (adding) composed units was allowed in proportional situations.

Attribute 1C: Making a multiplicative comparison within measure spaces

Another way to reason proportionally derives from a second conception of ratio, namely as a multiplicative comparison of two quantities. In general, a multiplicative comparison is formed by asking "How many times as great as one value is another?" or "What portion or fraction of one value is another?" (Thompson, 1994). This differs from an additive comparison (a non-ratio) which is formed by asking "How much greater is one thing than another" or "How much less is one thing than another?" For example, suppose the heights of two boxes are 9 in. and 6 in. We can compare the heights *multiplicatively* by saying that one box is 1 ½ times as tall as the other or that one is 2/3 the height of the other. Alternatively, we can compare the heights *additively* by saying that one box is 3 in. shorter than the other or that one box is 3 in. longer than the other. If a multiplicative comparison remains constant as both quantities co-vary, then the situation is proportional. Sub-attributes 1C and 1D examine comparisons that are made within measure spaces (e.g., heights to heights) and across measure spaces (e.g., heights to lengths), respectively. Multiplicative comparisons can extend to whole numbers, unit fractions, proper fractions, and improper fractions.

It can be difficult to infer when a teacher has formed a multiplicative comparison as opposed to simply finding a factor that relates two values (e.g., by division). An indicator of a multiplicative comparison is the presence of quantitative reasoning, through a verbal description of comparing two quantities relatively, and through the interpretation of meaning of the resulting quotient in context.

Example. Reconsider the nursing problem from sub-attribute 1A:

> *A drug comes in packets of 120 mg diluted in 2 ml of fluid. How much diluted drug should be administered for a dose of 300 mg?*

A solution path using a multiplicative comparison within measure spaces would involve the teacher asking herself, "How many times as great as 120 mg is 300 mg?" and then determining that 300 is 2.5 times 120. The teacher could then use her understanding of the invariance of this multiplicative comparison in proportional situations to find 2.5 x 2 ml, which is 5 ml.

Limited Understandings. We have identified the following limited understandings of this sub-attribute from teachers in the clinical interviews:

- Some teachers were able to form multiplicative comparisons only for whole number factors or for "easy" factors. For example, in comparing 2 and 5, a teacher figured that 5 is 2 ½ times greater than 2, but he had difficulty figuring out what portion 2 is of 5.
- Teachers may find the correct factor linking a source and target number and use that factor with the other quantity in the ratio, but the factor appears to have been produced calculationally (e.g., by dividing) and without explicit evidence of quantitative reasoning (i.e., comparing how many times as large one quantity is than the other).
- Additionally, teachers who find a factor, but who have not formed a multiplicative comparison, may have trouble interpreting the meaning of their factor (or quotient) in context, may have trouble with the order of division, or may have trouble with the interpretation of division in the context.

Attribute 1D: Multiplicative comparison across measure spaces

With an understanding of this sub-attribute, the teacher can form a ratio as a multiplicative comparison across measure spaces and understand that the situation is proportional if the multiplicative comparison holds for all corresponding values of w and z. This multiplicative comparison (of z to w) is the constant of proportionality, k, by which $wk = z$ for all corresponding values of w and z. The teacher consistently uses the constant of proportionality to support reasoning about proportions.

What separates the formation of a multiplicative comparison from simply finding a factor is the presence of quantitative reasoning. One aspect of this quantitative reasoning is the relative comparison of two quantities. In the case of the formation of a multiplicative comparison across measure spaces, another aspect of quantitative reasoning is what we call *reasoning with units.* Specifically, the teacher appears to understand that when one forms a multiplicative comparison across measure spaces, the comparison is between the *values* of the quantities, not the quantities themselves. For example, it does not make sense to say how many times as much distance is there as time, but one can form a ratio to compare the *amounts* of distance and time.

Example. We return to the nursing problem one more time:

> *A drug comes in packets of 120 mg diluted in 2 ml of fluid. How much diluted drug should be administered for a dose of 300 mg?*

This problem can also be solved by forming a multiplicative comparison of the measure of the ml of fluid (2 ml) to the measure of the mg of the drug (120 mg). Specifically 2 is 1/60 of 120. To complete the problem, the teacher needs to maintain the same ratio of dilution by finding 1/60 of 300, which is 5. Finally, the teacher interprets the 5 in terms of the context: the 5 refers to 5 ml of fluid. Notice that it doesn't make much sense to ask, "How many times as much fluid is there as drug?" We drop the units for the quantities, carry out the multiplicative comparison with the values of the quantities, and then reinterpret the result in terms of the context at the end. This often happens with across-measure space reasoning because two different types of units are being compared.

Limited Understandings. The limitations are the same as for sub-attribute 1C.

Attribute 2: Connections between Ratios and Fractions

This attribute involves making conceptual links between ratios and fractions. For many middle grades teachers, the topics of ratios and fractions are compartmentalized rather than connected by an underlying form of reasoning. This may be in part because textbooks isolate ratios from fractions (Clark, Berenson, & Cavey, 2003). Furthermore, because ratios are typically expressed in fraction form in textbooks (e.g., as *a/b*), many teachers believe that "ratio" is just another name for "fraction" without realizing that ratios and fractions, although connected conceptually, do not have identical meanings (Clark, et al., 2003). Therefore, this attribute is important because teachers need to understand the relationships between ratios and fractions so that they can facilitate the development of appropriate connection-making in children. Although there are many connections among rational numbers and operations, we focus on three in this section—reinterpreting a ratio as a fraction (with two different meanings) and differentiating between fraction and ratio operations and their quantitative meanings—because of their prominence in our interviews with teachers.

Attribute 2A: Using composed unit reasoning to reinterpret a ratio as a fraction

The teacher with this sub-attribute understands that a ratio can be reinterpreted as fraction by treating the ratio as a composed unit, partitioning it to obtain a unit ratio, and then interpreting the unit ratio as a fraction. The teacher is able to reinterpret a given ratio in relationship to some appropriate whole. Such reasoning also supports the partitive meaning of division, which answers the question "How much of one quantity is associated with one unit of a second quantity?"

Example. This example is drawn from Lobato and Ellis (2010).

Consider a salad dressing that is 2 parts vinegar to 5 parts oil. The ratio of vinegar to oil can be expressed as 2:5. We seek to reinterpret this part-part comparison as a part-whole comparison, that is, as the fraction 2/5. To do this, we ask the question, "2/5 is two-fifths *of what*?" The vinegar and oil can be joined as a composed unit (Attribute 1A) to form one batch of salad dressing. The ratio 2:5 will be maintained if the batch is partitioned into five equal parts. Because we are operating on a composed unit, we partition both the oil and the vinegar into five equal parts. Partitioning the oil yields 1 of the original 5 parts of oil. Partitioning the vinegar is a little more involved. One can split both parts of vinegar into five equal portions and take one 1/5 from each of the original parts of vinegar (see Figure 3). Understanding that this yields 2/5 part vinegar in each portion relies on interpreting the fraction 2/5 as two one-fifths. Consequently, reinterpreting the ratio 2:5 as the fraction 2/5 means that a recipe of this salad dressing made with 1 part oil has 2/5 of one part of vinegar. In sum, this reinterpretation of the ratio 2:5 as a fraction relies on thinking of the ratio as a composed unit (sub-attribute 1A).

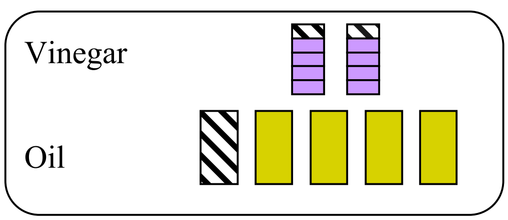


Figure 3. One-fifth of the batch is 2/5 parts vinegar and 1 part oil.

Limited Understandings. We were surprised to find that making connections between ratios and fractions was difficult for the teachers we interviewed, even those with the strongest mathematics backgrounds.

- Some teachers implicitly treated ratios as identical to fractions symbolically and did not reinterpret the meaning of a ratio as a fraction.
- Many teachers were unable to locate an appropriate whole that would allow them to reinterpret the ratio a:b as the fraction a/b of some whole.
- Other teachers conceived of ratios and fractions as separate and thought it inappropriate to reinterpret a ratio as a fraction because fractions only express part-whole relationships while ratios can express part-part or part-whole relationships.

Attribute 2B: Using a multiplicative comparison to reinterpret a ratio as a fraction

The teacher with this sub-attribute understands that a ratio can be reinterpreted as fraction by treating the ratio as a multiplicative comparison across measure spaces. Specifically the teacher realizes that a ratio $x:y$ can be interpreted to mean that x is x/y of y for each corresponding pair of x and y values.

Example. Reconsider the same salad dressing that is 2 parts vinegar to 5 parts oil. There is a second way to reinterpret the ratio 2:5 as a fraction. We begin by asking the same question, "2/5 is two-fifths *of what*?" Because the ratio 2:5 does not indicate the exact amounts of vinegar or oil used in a particular recipe, the dressing could use 4 cups of vinegar and 10 cups of oil, 6 tablespoons vinegar and 15 tablespoons oil, ½ pint vinegar and 1¼ pints of oil, and so on. For each associated amount of oil and vinegar, we make a multiplicative comparison across measure spaces, by asking, for example, "4 is what portion of 10?", or "6 is what portion of 15?" The meaning of 2/5 as a fraction in this scenario is that, in each case, there is 2/5 as much vinegar as oil. In sum, this reinterpretation of the ratio 2:5 as a fraction relies on thinking of the ratio as a multiplicative comparison (sub-attribute 1D).

Limited Understandings. In addition to the limitations that we identified for sub-attribute 2A above, many teachers were unable to identify one of the quantities in the ratio as an appropriate whole to use when reinterpreting a ratio (e.g., in the ratio 2:5, 2 is 2/5 of 5).

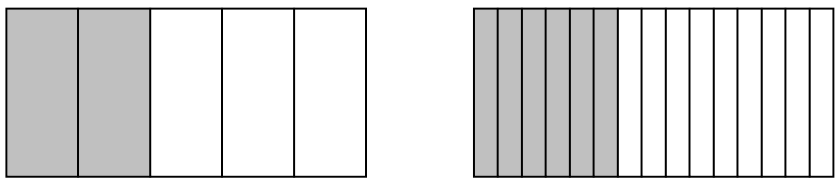
Attribute 2C: Differentiating fraction and ratio operations

Understanding how arithmetic operations with ratios can differ from operations with fractions is the third important aspect of the connection between ratios and fractions. For example, Mochon (1993) discussed examples when fraction addition proceeds differently from ratio addition. Consider a basketball player who throws two sets of 8 free shots. In the first set he makes 3, and in the second set he makes 4. If we look at his total performance, his success is 7 out of 16 shots. Although this can be expressed in fraction form as 7/16, the ratio addition proceeded as (3:8) "+" (4:8) = (7:16), rather than as 3/8 + 4/8 = 7/8. During interviews with teachers, we found that because ratios are represented symbolically as fractions, teachers invoked rules related to fractions (such as forming equivalent fractions, multiplying by a fraction equivalent to one, or adding fractions) without reinterpreting the meaning of the ratios as fractions or thinking about the "wholes" in the particular context.

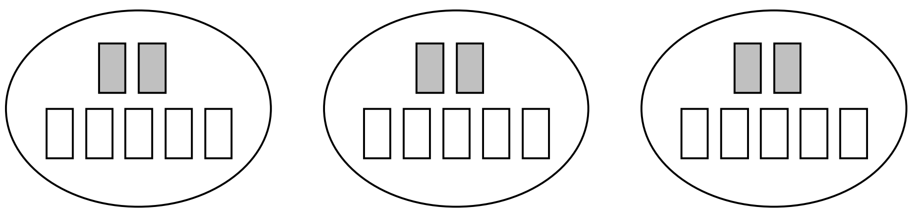
Example: The following problem emerged during an interview with one teacher, and was perplexing for other teachers to whom we posed the dilemma:

During professional development, one teacher shared with a colleague:

I'm confused about something. Suppose that you have a small batch of salad dressing made from 2 parts vinegar and 5 parts oil. A typical problem is to ask students, how much vinegar we will need if a larger batch of the salad dressing has 15 parts oil. If can show students that the answer is 6 parts vinegar by thinking of 2/5 and 6/15 as equivalent fractions. Then the amounts represented by both are the same and 2/5 x 3/3 = 6/15 because 3/3 = 1.



But if I think about 2:5 as a ratio and triple the batch of salad dressing, I get a lot more salad dressing, and I think I'm multiplying by 3 not 3/3.



What would you say to this teacher?

Several understandings are involved when interpreting these drawings:

- In the top drawing, if the rectangles both represent the same whole, then multiplying 2/5 by 3/3 means that quantitatively you are splitting each 1/5 into 3 equal parts (for a total of 15 parts) and each of the two 1/5s into 3 equal parts (for a total of 6 parts).
- If the "wholes" for both 2/5 and 6/15 are the same, then the amounts represented by 2/5 and 6/15 will be the same. In contrast, if both quantities in the ratio 2 parts vinegar to 5 parts oil are tripled, the batch or total amount of salad dressing is tripled.
- In the top drawing, one can perform fraction operations, i.e., 2/5 x 3/3 = 6/15, but in the bottom drawing, one is performing a ratio operation (2/5 + 2/5 + 2/5 = 6/15).
- The fraction and ratio drawings can be connected in several ways:
  - Consider the 5 boxes representing 5 parts oil in the bottom picture as the whole (and as the leftmost rectangle in the top drawing). Then the vinegar (2 parts) is 2/5 of the whole (2 is 2/5 of 5). Now consider all 15 boxes representing 15 parts oil in the bottom picture as the whole (as the rightmost rectangle in the top drawing). Then the vinegar (6 parts) is 6/15 or 2/5 of the whole (15 parts). This interpretation assumes that the rectangles at the top represent different "wholes," but the 2/5 is constant as 2/5 of the whole. (This is difficult for teachers because they often make an implicit assumption that the rectangles in the top picture represent the same amount).

- Alternatively, if we want the rectangles at the top to represent the same whole, then what whole is that? It could be one part oil. Then 2/5 of one part is the amount of vinegar that is needed to satisfy the recipe, and 6/15 of one part is the same *amount* of vinegar. The link to the bottom drawing is that you can take the representation of 2 parts vinegar to 5 parts oil and partition both quantities into 5 equal parts (as shown in Figure 3 above), to arrive at 2/5 of a part of vinegar being associated with each part of oil. Similarly you can take the representation of 6 parts vinegar to 15 parts oil and partition both quantities into 15 equal parts, to arrive at 6/15 or 2/5 of a part of vinegar being associated with each part of oil.

Limited Understandings. The situation in the example above was very challenging for teachers to make sense of in the interviews. Perhaps teachers focused on the numerator and denominator as numbers that were both increasing rather than quantities. As a result, they did not appear to realize that multiplying by 3/3 results in partitioning (rather than iterating) the quantity represented in the top drawing. Additionally teachers had difficulty linking the two representations on the page because they were not able to locate an appropriate "whole" in the ratio drawing. Specifically, they had trouble seeing the 15 small rectangles in the bottom drawing as a whole and interpreting the shaded rectangles as part of that whole.

Attribute 3: Appropriateness

We want teachers to determine whether or not a situation is proportional by looking for a "many to one" (or "many to some") relationship that continues as the two quantities co-vary. Too often adults as well as children rely on the following types of superficial cues to decide whether or not a situation is proportional: (a) if there are three numbers given and one number missing; or (b) if the situation involves key words such as "per," "rate," or "speed." Furthermore, teachers often misinterpret situations involving indirect proportions or situations that can by modeled as $y = \mathrm{m}x + b$ as being proportional.

Example. Consider the following task, from Cramer, Post, and Currier (1993):

> *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie had completed 15 laps, how many laps had Sue run?*

According to Cramer, et al. (1993), 32 out of 33 preservice elementary teachers in a mathematics methods class inappropriately set up the following proportion to solve the problem: $9/3 = x/15$ and arrived at an incorrect answer of 45 laps. When two quantities are related proportionally, they are in a "many-to-one" relationship that holds across values. In the running context, there is also a "many-to-one" relationship: We can think about Sue as having run 3 laps for each lap that Julie ran because $9 = 3 \cdot 3$. However, this relationship does not continue. When Sue runs one more lap, so does Julie, which means that Sue will have run 10 laps when Julie has run 4. The situation is linear but not proportional. Consequently, when Julie has run 15 laps, Sue will have run $15 + 6 = 21$ laps.

Limited Understandings. Research from the *Does It Work* (DiW) project and from our DTMR interviews suggests that being able to recognize a situation as proportional is pivotal to proportional reasoning and is more difficult for teachers than we expected. If the context explicitly mentions addition, then most teachers seem to realize that the situation is not proportional. Problems that do not have such clear clues are harder. Teachers often misinterpret situations involving inverse proportions as proportional as well as problems set in a motion contest such the track problem described above. The DiW teachers could readily recognize a typical missing value problem as proportional when stated as an across-measure spaces problem. Linking within-measure space reasoning to the proportion algorithm was harder for some teachers. The fact that we observed some teachers reasoning additively on a problem involving blocks on a balance scale suggests teachers, like students, may find geometric problems and problems involving scaling or similarity the most difficult to recognize as proportional.

Attribute 4: Ratios in Context as a Network of Related Quantities

Comprehending quantitative situations involving ratios can be seen as relating two perspectives: (a) a measurable quantity (e.g., speed) can be experienced directly (e.g., a teacher can *feel* when he or she walks faster); and (b) the quantity can emerge when one constructs a ratio between two quantities (e.g., distance and time) (Confrey & Smith, 1995; Noble, Nemirovsky, Wright, & Tierney, 2001; Piaget, 1970; Thompson, 1994). If these two perspectives are coordinated, then following four quantities form a network of quantities: the two quantities from which the ratio is formed, the emergent ratio as a single holistic entity (not just a coordination of two quantities), and a quantity (such as speed) that measures an attribute in the real world situation. For example, the ratio of the height of a wheelchair ramp to the length of its base is a measure of the steepness of the ramp. Four quantities are related—height, length, the ratio of height to length as a single entity, and steepness.

The ability to conceive of a ratio in context as a network of related quantities involves the following three sub-attributes: (a) forming a ratio-as-measure, (b) interpreting the meaning of equality of ratios, and (c) making connections with linear functions. In the wheelchair ramp situation described previously, forming the ratio of the height of the ramp to the length of its base and understanding that the ratio measures the attribute of the steepness of the ramp is called forming a "ratio-as-measure." Rather than treating ratios only as representing numerical relationships, "ratio-as-measure" links a ratio to what it measures in a real-world context. The notion of ratio-as-measure can be extended to recognize that a set of equivalent ratios can measure the same intensity of a given attribute. Consider the wheelchair ramp again. Suppose that we have a ramp with a height of 1 ft and base of 12 ft. There are infinitely many ramps with the same degree of steepness (e.g., a ramp with a height of 6 in and base of 72 in; 3 ft and 36 ft; 8 in and 96 in; and so on). Furthermore, when we set any two of these ratios equal in a proportion (e.g., 6/72 = 8/96), an important part of understanding the meaning of equality in this situation is that the two ratio measure the same intensity of the attribute being measured (here, steepness). Finally, a set of infinitely many equivalent ratios can be expressed as a linear function of the form $y = mx$, and the meaning of m can be interpreted as the slope of the function and as a ratio-as-measure. In the wheelchair example, if x is the length of the base of a ramp and y is the ramp's height, then the proportional relationship between the lengths and bases for ramps with the same steepness can be expressed by $y = 1/12x$, where 1/12 is a ratio-as-measure of the steepness of the ramp. Finally, the slope of 1/12 can be interpreted in the context in two ways (using ideas from Attribute 2): (a) the base of one of these ramps is always 1/12 the height; or (b) for a ramp with a base of length 1 unit (say 1 ft), then the height will be 1/12 ft, in order to preserve the steepness.

Attribute 4A: Forming a ratio-as-measure

A teacher with this understanding interprets the emergent quantity of a ratio in context as the measure of some attribute in a real-world situation. For example, the ratio of orange concentrate to water is a measure of the oranginess of the juice. The ratio of number of miles traveled to amount of gasoline used is a measure of *gas efficiency*. A ratio that measures some attribute is referred to as a "ratio-as-measure" (a term coined by Simon and Blume, 1994). Forming a "ratio-as-measure" involves two non-numerical processes: (a) isolating the attribute of interest from other attributes in the situation and (b) determining of the effect of changing various quantities on the attribute of interest (Lobato, 2008; Olive & Lobato, 2008). Forming a ratio-as-measure involves a network of related quantities. In the gas efficiency example, the relevant quantities are distance traveled, amount of gasoline used, the formation of the ratio of distance traveled to amount of gasoline as a reified entity, and quantity of gas efficiency.

Example. In a study with preservice elementary teachers, Simon and Blume (1994) posed the ratio-as-measure task shown in Figure 4. Instead of asking teachers to calculate the slope of the ski ramp, Simon and Blume focused on the creation of a measure for steepness. They found that most of the preservice teachers preferred the "height of the ramp minus the length of the base" as a measure of the steepness of the ramp than the "height divided by the length of the base."

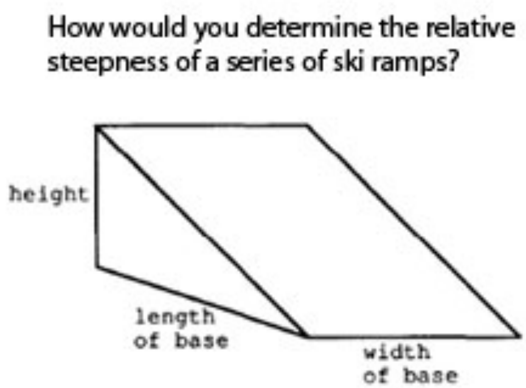


Figure 4. Simon and Blume's ski ramp task.

Limited Understanding. We found that two limited understandings were common in our interviews with teachers. First, some teachers coordinated quantities in a static manner rather than forming a holistic structure. Thus, they might say that a ratio measures the "amount of vinegar to oil" in a salad dressing recipe rather than the "taste" or "vinegary-ness." Other teachers expressed the belief that reasoning about taste (or some other attribute) is not essential or is superfluous to understanding the mathematical idea.

Attribute 4B: Interpreting the meaning of equality of ratios

A teacher with this understanding interprets the meaning of equality of two ratios (such as *a/b*= *c/d*) as indicating that both ratios measure the same intensity of the attribute being measured in the situation (such attributes may include speed, taste, density, gas efficiency, etc.) (Harel, Behr, Lesh, & Post, 1994). Furthermore each ratio is conceived of as an instantiation of a single rate, and the equal sign is conceived of relationally as meaning that each side of the equation has the same value (Kieran, 1981; Knuth, Stephens, McNeil, & Alibali, 2006). Thus, a sophisticated understanding of the equality of the ratios a/b and c/d involves a complex network of related quantities—*a*, *b*, *c*, *d*, the ratio *a/b* as a quantity, the ratio *c/d* as a quantity, and some attribute being measured in the situation (such as speed or taste). In a limited understanding of the equality of two ratios, a teacher may only deal with four (extensive) quantities— *a*, *b*, *c*, and *d*,— coordinating them by reasoning that "*a* is to *b* as *c* is to *d*"; thus, failing to form the ratios *a/b* and *c/d* as entities in their right, whose values can be the same and can be seen as measuring the same intensity of some attribute in the situation.

Example. Consider the following proportion that arises out of a salad dressing context similar to the one presented above: $\frac{2}{3} = \frac{10}{15}$. *What does the equal sign mean in this context, namely what is equal?*

The equal sign indicates that the two batches of salad dressing (one made from 2 parts vinegar and 3 parts oil and the other made from 10 parts vinegar and 15 parts oil) are equally "vinegary," meaning that they will taste the same. To understand why this is the case, we reinterpret the ratios 2:3 and 10:15 as fractions using composed units (using sub-attribute 2A). One meaning of 2:3 as the fraction 2/3 is that a recipe made from 2/3 cup vinegar and 1 cup oil will maintain the 2:3 ratio, thus preserving the taste. Similarly, the ratio 10:15 can be reinterpreted as a fraction: A salad dressing made from 10/15 cup vinegar and 1 cup oil will preserve the 10:15 ratio and will taste the same as the original recipe. To establish that 2/3 cup is equal to 10/15 cup, we can partition a full cup into 3 parts and then partition each of those parts into 5 smaller parts. This creates three equal parts, each containing 5/15 cup. Each group of 5 one-fifteenths is equal to 1/3 cup which implies that 2/3 cup (1/3 + 1/3) is equal to 10/15 cup (5/15 + 5/15).

Limited Understanding. In our interviews, we found that our participants were often unable to articulate a meaning of the equal sign in a proportion. For those teachers who did articulate a meaning, we found the following limited conceptions:

- Some teachers conceived of one ratio as a whole number iterate of the other, i.e., one ratio is a "subset" of or is "contained" in the other.
- Others implicitly treated ratios as identical to fractions (symbolically), without a conceptual reinterpretation from ratios to fractions, and consequently applied rules for obtaining and simplifying equivalent fractions.
- Some teachers focused on operations to each quantity in the ratio without saying what was equal and without treating the ratio as an entity.
- Equality was also conceived as a "comparison point," e.g., it prompted some teachers to compare two ratios. For example, in the statement "2 is to 5 as 6 is to 15," the "as" was understood to mean "the same as." This conception allowed teachers to keep both

quantities separate instead of conceiving of the ratio as a single quantity. This may play into the notion of fractions and ratios as separate.

Attribute 4C: Making connections with linear functions

A teacher with this understanding can conceive of $y = mx$ as a statement of proportionality where corresponding values of x and y are related multiplicatively by m, the *slope* of the function. Indeed, Karplus, Pulos, and Stage (1983) characterized proportional reasoning as "a term that denotes reasoning in a system of two variables between which there exists a linear functional relationship" (p. 219). Connecting proportionality with linear functions (of the $y = mx$ form) involves understanding a network of related quantities—the quantity of the independent variable, the quantity of the dependent variable, the formation of a ratio of independent to the dependent variable (i.e. $y/x = m$), and the interpretation of m as measuring some quantity in the real world situation.

Example. Consider the following task for teachers:

> Your school district material introduces algebra by asking students to write equations for proportional situations such as the following:
>
> > *A cola recipe uses 1 ½ ml of cinnamon oil for every 3 ½ ml of orange oil. Write an equation to find the amount of cinnamon oil that is needed, given any amount of orange oil.*
>
> How would you like students to approach this problem?

A teacher with an understanding of sub-attribute 4C might first think about the relationship between the amount of cinnamon oil and the amount of orange oil. This could result in the formation of a multiplicative comparison between the two quantities. Specifically, the teacher might ask herself, "1 ½ ml is what part of 3 ½ ml"? Three and ½ ml can be conceived of as 7 halves, and 1 ½ is 3 halves. Thus, the amount of cinnamon oil is 3/7 the amount of orange oil. Because this is a proportional situation, the relationship holds for any amount of cinnamon oil (y) and an associated amount of orange oil (x). Hence $y = 3/7x$.

Limited Understandings. We identified the following limited understandings of this sub-attribute from teachers in the clinical interviews and from prior research on algebraic reasoning:

- Several teachers didn't think it was necessary to have an x and a y in an equation representing a proportion. For instance, one teacher appeared to form a composed unit in a proportional situation (e.g., a 10:4 unit in a speed situation where 10 represents 10 cm and 4 is 4 sec) and then reasoned that she could maintain the proportional relationship by multiplying each quantity by the same number (x). Thus, she expressed her equation as $10x = 4x$.
- Some teachers arrived at a correct equation of the form $y = mx$, but did so using the proportion algorithm. For example, for the Cola Problem above, a teacher may set up the proportion $3.5/1.5 = x/y$ and then manipulate the proportion algebraically to arrive at the

equation $3.5y = 1.5x$ and then $y= 1.5/3.5x$ and finally $y = 3/7x$ (where x is the amount of orange oil and y is the amount of cinnamon oil) but not be able to interpret the meaning of 3/7 in the cola situation.

- Teachers, like students, may also experience difficulties interpreting the meaning of literal symbols, e.g., treating them like abbreviations or labels rather than as varying quantities (Kilpatrick, Swafford, & Findell, 2001; MacGregor & Stacey, 1997). For example, in the Cola Problem, a teacher may erroneously report the equation as 3/2C = 7/2O, and treat C and O as labels (e.g., C for cinnamon) rather than as amounts.

References

Ben-Chaim, D., Keret, Y., & Ilany, B. (2007). Designing and implementing authentic investigative proportional reasoning tasks: the impact on pre-service mathematics teachers' content and pedagogical knowledge and attitudes. *Journal of Mathematics Teacher Education, 10*, 333-340.

Clark, M. R., Berenson, S. B., & Cavey, L. O. (2003). A comparison of ratios and fractions and their roles as tools in proportional reasoning. *Journal of Mathematical Behavior, 22*, 297-317.

Confrey, J., & Smith, E. (1995). Splitting, covariation, and their role in the development of exponential functions. *Journal for Research in Mathematics Education, 26*(1), 66-86.

Cramer, K., Post, T., & Currier, S. (1993). Learning and teaching ratio and proportion: Research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159-178). New York: Macmillan.

Ginsburg, H. (1997). *Entering the child's mind*. Cambridge, UK; Cambridge University Press.

Harel, G., Behr, M., Lesh, R., & Post, T. (1994). Invariance of ratio: The case of children's anticipatory scheme for constancy of taste. *Journal for Research in Mathematics Education, 25*(4), 324-345.

Hoyles, C., Noss, R., & Pozzi, S (2001). Proportional reasoning in nursing practice. *Journal for Research in Mathematics Education, 32*(1), 4-27.

Hull, L. S. H. (2000). *Teachers' mathematical understanding of proportionality: Links to curriculum, professional development, and support*. Unpublished doctoral dissertation, The University of Texas at Austin, Austin, TX.

Kaput, J. J., & Maxwell-West, M. (1994). Missing-value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 235-287). Albany, NY: State University of New York Press.

Karplus, R., Pulos, S., & Stage, E. (1983). Proportional reasoning of early adolescents. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematical concepts and processes* (pp. 45-90). New York: Academic Press.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317-326.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.) (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*(4), 297-312.

Lamon, S.J. (1994). Ratio and proportion: Cognitive foundations in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 89-120). Albany, NY: State University of New York Press.

Lamon, S.J. (1995). Ratio and proportion: Elementary didactical phenomenology. In J. T. Sowder & B. P. Schappelle (Eds.), *Providing a foundation for teaching mathematics in the middle grades* (pp. 167-198). Albany, NY: State University of New York Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In K. Lester Jr. (Ed.), *Second handbook of research on*

mathematics teaching and learning (pp. 629–667). Charlotte, NC: Information Age Publishing.
Lesh, R., Post, T., & Behr, M. (1988). Proportional reasoning. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.
Lobato, J. (2008). When students don't apply the knowledge you think they have, rethink your assumptions about transfer. In M. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 289-304). Washington, DC: Mathematical Association of America.
Lobato, J., & Ellis, A. B. (2010). *Essential understandings: Ratios, proportions, and proportional reasoning.* In R. M. Zbiek (Series Ed.), *Essential understandings*. Reston, VA: National Council of Teachers of Mathematics (NCTM).
MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33*(1), 1-19.
Mochon, S. (1993). When can you meaningfully add rates, ratios, and fractions? *For the Learning of Mathematics, 13*(3), 16-21.
Noble, T., Nemirovsky, R., Wright, T., & Tierney, C. (2001). Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education, 32*(1), 85-108.
Noelting, G. (l980). The development of proportional reasoning and the ratio concept. Part 1: Differentiation of stages. *Educational Studies in Mathematics, 11*, 217–253.
Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In M. K. Heid & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics: Research syntheses* (pp. 1-54). Charlotte, NC: Information Age Publishing, Inc. and the National Council of Teachers of Mathematics.
Piaget, J. (1952). *The child's conception of number*. London: Routledge and Kegan Paul.
Piaget, J. (1970). *The child's conception of movement and speed*. New York: Basic Books.
Simon, M.A., & Blume, G.W. (1994). Mathematical modeling as a component of understanding ratio-as-measure: A study of prospective elementary teachers. The Journal of *Mathematical Behavior, 13*(2), 183-197.
Smith, J. P. & Thompson, P. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (p. 95-132). New York, NY: Lawrence Erlbaum Associates.
Thompson, P.W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: State University of New York Press.
Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 127–174). New York: Academic Press.
Vergnaud, G. (1988). Multiplicative Structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in middle grades* (pp. 141–161). Reston, VA: National Council of Teachers of Mathematics; Hillsdale, NJ: Erlbaum.