Longitudinal Learning of Viable Argument in Mathematics for Adolescents (LLAMA)
Assessment Version A
Master Scoring Rubric
5.6.19

Read Correctly Rubric

The "read correctly" score applies to all items. Each item is scored with its own rubric for the argument score. See next pages for those.

The general goal of the argument score (in the rest of the rubric) is to measure the student's understanding of LLAMA argumentation ideas and practices. However, if the student doesn't understand some of the mathematical terms in an item, or if they don't understand what the item is asking them to do, then that can interfere with their ability to display their knowledge of the argumentation concept(s) we are seeking to measure. The purpose of the *read correctly* score is to measure whether the student had access to the prompt. If a student scores a 2 on *reads correctly*, then we are confident that their argument score on the item accurately reflects their understanding of the targeted argumentation concept. They could still get a 0 on the argument score but if they do, it is because of their low argumentation reasoning (for instance, maybe they provide an empirical argument for a general claim). If a student scores a 0 on *reads correctly*, then we have no evidence that they would be able to demonstrate their understanding of the targeted argumentation concept on the item, even if they understood that argumentation concept robustly (for instance, maybe they understand indirect arguments thoroughly, but we can't tell because they so badly misinterpreted the task instructions). Thus, the *read correctly* score is independent of the student's understanding of the targeted argumentation concept.

Purpose: The *read correctly* score measures the degree to which the student has shown an understanding of

(a) the mathematical objects and definitions in the item, and

(b) the task's format, structure, and instructions (e.g., develop an argument, assess the claim, critique the presented argument)

that would enable the student to demonstrate their understanding of (c) the argumentation concept(s) we seek to assess with the item (e.g., what is a counterexample, when is an exhaustion argument appropriate, etc.)

Category / Rating	0	1		2
Read correctly (understand situation)	Respondent shows no evidence of understanding either (a) or (b) above. OR The understanding of (a) and (b) demonstrated is not sufficient for the respondent to be able to score higher than a 0 on the argument score, even if they understood the argumentation concept in the task.	Respondent demonstrates some understanding of (a) or (b) above, but it is not clear that they understand these fully enough that their argument score accurately reflects their understanding of the argument concept in the task. In other words, you could imagine someone with this understanding of (a) and (b) getting at least a 1 on the overall score, but it is not clear that anyone with this understanding of (a) and (b) would be able to get a 3.		Respondent appears to understand (a) and (b) fully enough that the argument score accurately reflects their understanding of the argument concept in the task.
	Code: **NR, N**	Code: **U**	Code: **P**	Code: **Y**

Problem 1

A student in your class claims that the sum of any three consecutive counting numbers is divisible by 3.

a) Develop a viable argument for or against the student's claim.
b) Explain why your argument in part a) is viable.

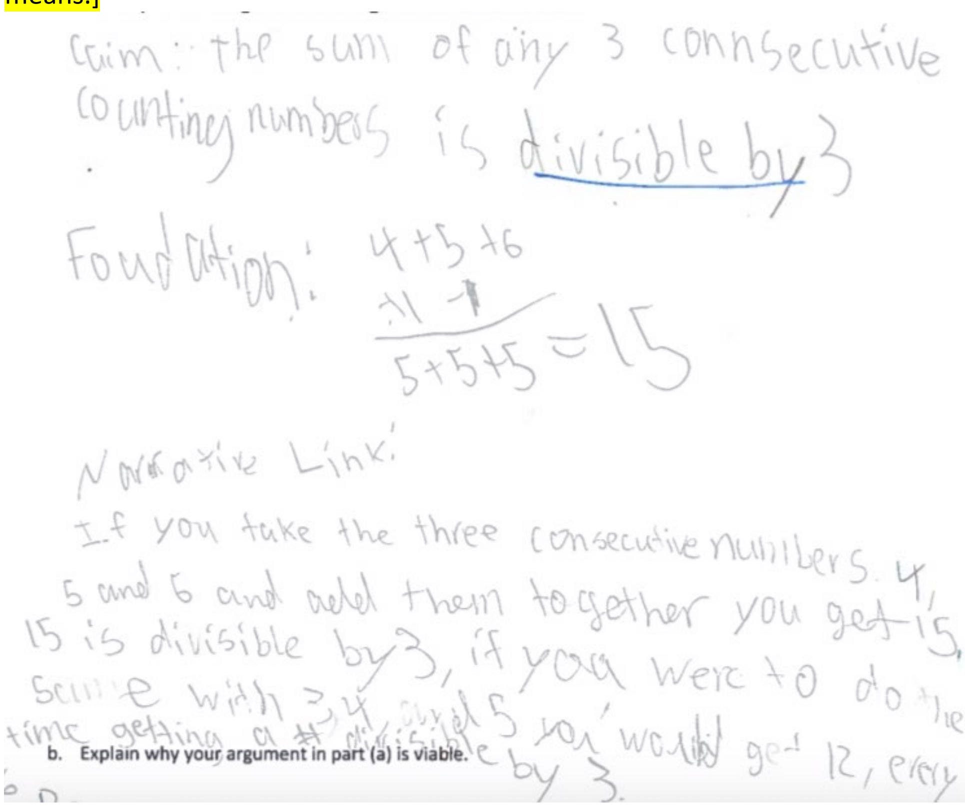
Purpose of item: Item assesses respondents' ability to recognize a true generalization and develop an appropriate general argument. A variety of argument types (e.g., variable, generic example, recursion, though experiment, indirect, etc.) may be accepted as viable.

Rating	Description
3: Viable Argument	If response fully meets the purpose of the item, it should be scored a 2 or a 3. Response includes all of the following: (1) An assertion that the students' claim is correct. (2) A clear assertion of generality either in an explicit claim or explicit in the support of the claim already given in the task. (3) Sufficient detail to infer that the respondent is arguing for all sums of three consecutive numbers. (4) A representation, instantiation, or narrative that **links** the conditions (*sum of three consecutive counting numbers*) to the conclusion (*can be divided by three*) by appealing to the structure of "consecutive" and "sum," or the response has some canonical general representation (e.g., variable equation showing a viable transformation or expression) that serve as a symbolic expression of the meaning of the work, which a narrative link might otherwise express. **Example Claims:** • All sums of three consecutive numbers are divisible by 3 • Sums of three consecutive are divisible by 3 • The student is correct. **Example response 1 [Variable-based argument]:** n + n+1 + n+2 = 3n + 3 = 3 (n + 1) [or could be left as 3n + 3—it is okay if they don't explicitly factor out the 3]. (Or, n-1+n+n+1=3n, or, n-2+n-1+n.)

Example response 2 [Recursive argument]: 1 + 2 + 3 = 6. 2+3+4-(1+2+3)=1+1+1. 6+3=9. 9/3=3. The very first set of three consecutive counting numbers is 1+2+3, which adds to 6. So, any set after that would be adding 3, because each number in the set increases by 1 with each new set, therefore maintaining the divisible nature of the problem.

- Note that there are other types of recursive arguments that are fully viable, such as: you can always get from one sum to the next by taking the first of the three numbers, adding 3 to it, and making it the last number (of the new three consecutive numbers).

Example response 3 [Generic example/diagram]: [A diagram or generic example illustrating the structure expressed in Example response 1, and acknowledges generality.] [Student may not completely articulate what "do the same every time" means.]


Claim: the sum of any 3 connsecutive counting numbers is divisible by 3

Foundation: 4+5+6
+1 -1
5+5+5=15

Narrative Link:
If you take the three consecutive numbers 4, 5 and 6 and add them together you get 15, 15 is divisible by 3, if you were to do the same with 3, 4, and 5 you would get 12, every time getting a # divisible by 3.

b. Explain why your argument in part (a) is viable.

	Example response 4 [Natural language]: A natural language response that explicitly describes the structure in example response 1 (e.g., Three consecutive numbers can be written as a middle number and the middle number minus 1 and the middle number plus 1. This sums to 3 times the middle number because the minus 1 and plus 1 cancel.)
2: Elements of Viable Argument	Response includes all of the following: (1) An assertion that the student's' claim is correct. (2) A clear <u>assertion of generality</u> either in an explicit claim or explicit in the support of the claim already given in the task. (3) Sufficient detail to infer that the respondent is arguing for all sums of three consecutive numbers. It is in the next feature that a Score of 2 is distinct from a 3. (4) A conceptual insight* that is appropriate for viable argument is present; however it is not represented, instantiated, or described in a sufficient manner. Perhaps, for example, a student describes a process but does not show explicitly that the process works for all possible types of cases. They are leveraging the conclusions of a correct result (prior or new), but not showing how they are engaging that result's conditions. Another way to say this is that there is an unclear sub-conceptual insight. The conceptual insight is sufficiently express so that the reader does not have to infer significantly, but there still not enough detail for it to be fully viable. *L**<u>inks</u>** <u>the conditions</u> (*sum of three consecutive counting numbers*) to the <u>conclusion</u> (*can be divided by three*) by appealing to the structure of "consecutive" and "sum". In this case the structure of the *conditions* must be expressed, not just noticing something about the conclusion. **Example response 1 [recursion]:** 1+2+3=6; 2+3+4=9; and 3+4+5=12, and all the sums are multiples of 3. Each time you move up the consecutive counting number it adds 3. 3 is always divisible by 3. The consecutive counting numbers are always multiples of 3. **Example response 2 ["off set" structure]**: This is true because in every sequence there will always be a number divisible by 3, the offset of the other two numbers will always add up to 3, -3, or 0. This means that will always be divisible by 3. **[Note: Viable CI for "offset" but sub-CI needed to support "[sum of multiples of three] will always be divisible by 3]."** **Example response 3 [Recognize the generality and structural link between conditions and conclusion but the articulation of this link is somewhat lacking].**

	Yes! becase if I take the 1st Number of the scuqunce and the Last # of the Sensese they equle the middel # doubled exmaple's: 1 2 3 | 6 7 8 | 10 11 12 | 3 4 5 4÷2=2 | 14÷7=2 | 22÷11=2 | 8÷4=2 if the two out side ones double that in the inside that is 2 of the inside # pus the inside # is 3 so it is divs bil by 3 **Example response 4 [Cases] [Special note; the example below would have been a 3 had the student noted in the narrative link that foundation addressed cases such as 139+140+141, i.e., the case 9+10+11, where more than the one's digit is involved]:**

1. **A student in your class claims that the sum of any three consecutive counting numbers is divisible by 3. (An example of three consecutive numbers is 4, 5, and 6.)**

7, 8, 9 123, 124, 125

15 24 247 $\frac{124}{3\,)\,372}$

a. Develop a viable argument for or against the student's claim.

Claim: Any 3 consecutive counting #s are divisible by 3 when added together.

MF: 1 2 3 = 6
2 3 4 = 9
3 4 5 = 12
4 5 6 = 15
5 6 7 = 16
6 7 8 = 21
7 8 9 = 24
8 9 10 = 27
9 10 11 = 30
0 1 2 = 3

2
3
4
5
6
7
8
9
10
1

24 + 30 + 31
60
90

28 + 29 + 30 =

123 = 6
234 = 9
345 = 12
456 = 15
567 = 18
678 = 21
789 = 24
8 9 10 = 27
9 10 11 = 30
20 30

EL: Every # has to end with a number that is 0 or anything else up to 9. I took each of those 1digit to show that each set of 1digit consecutive counting #'s is divisible by 3. If you add on tens, hundreds, etc. you are adding that together 3 times (ex: 130+130+130=390). Divide by 3 you get that # just add on ones place (1+2+3 3)6 = 2) 132) any git work

b. Explain why your argument in part (a) is viable.

This is a viable argument because I have a claim, foundation, and explanation, I also show all my work.

	Example response 5 [Vague conceptual insight]: For all sums of three consecutives, one number in the sum is divisible by 3 and the other two sum to a number divisible by three. **Example response 6 [Mismatch between referent CI and Narrative]:** Referent: (n+n+1+n+2)/3= a whole number. Narrative: The generic example is a referent for all other cases. The equation can have numbers replacing the variables and still work. When three consecutive numbers are added, the sum is divisible by three because one number is divisible by three and the other two will add up to a number that is divisible. [**Both the referent and the narrative express a CI that can be leveraged; however the linking structure is not explicit.**]
1: Limited Elements of Viable Argument	Respondent: • Recognizes the infinite domain of the claim and acknowledges that examples do not prove all cases, or perhaps expresses uncertainty about whether the claim is true for all cases. • Asserts that the student's claim is false and offers a counterexample and included enough detail to reasonable assure the rater that the role of counterexample in mathematical argumentation is understood. • Recognizes the infinite nature of the claim and attempts to argue for the claim using an inappropriate or incorrect prior result (such as one about even and odd structure). • Treats the claim as (or changes the claim to) an existence claim and provides an example (even if the example uses an incorrect calculation or interpretation). This is not the same as empiricism, because they are not using an example as proving a general claim. Evidence of searching for a conceptual insight and/or general structure, but no clear statement of a leverageable conceptual insight. Irrelevant conceptual insight that is not leverageable to a viable argument or based on incorrect prior result. **Example response 1 [Acknowledges infinite domain and express skepticism of examples]:** The method works with numbers 1, 2, and 3; 2, 3, and 4; 3, 4, and 5; however, I don't believe that a rule like this can continue on infinitely. **Example response 2 [False counterexample with some features of the claim's conditions or conclusion. Must have some evidence that the student interacted with the content and is explicit that the claim is false or provided an example with their perceptions of the conditions and not the conclusion]:** There exists a counterexample. 4+ 5+ 6=16. 16/3=5.333…, which is not a whole number. Alternatively. "There exists a counterexample. 4, 5, and 6. 4/3=1.333… and 5/3=1.666…. Even though 6 can be divided by 3…4 and 5 cannot."

Example response 3 [Inappropriate prior result]: This claim is true. There exists no counterexample. (a+b+C)/3 = a whole number. Odd+even+odd is even. Even+odd+even is even. But both still gives you a number divisible by 3.

Example Response 3 [Searching for conceptual insight]: The insight about mean is not alone leverageable to a viable argument, but it does indicate a search for a CI, not just making observations about the results.

All sums of three consecutive numbers are divisible by 3

1, 2, 3 = 6 ÷ 3 = 2
4, 5, 6 = 15 ÷ 3 = 5
7, 8, 9 = 24 ÷ 3 = 8
102, 103, 104 = 309 ÷ 103
21, 22, 23 = 66 ÷ 3 = 22

mean = T+T+T ÷ number of terms.

the claim proves true because When finding a mean you will get the average witch is in the middle. because there are 3 terms When you divide the sum of them by three you get the mean (Because there is also the number of terms.) Of the sequence that is in the middle of the three terms.

	b. Explain why your argument in part (a) is viable. Because to average a set of numbers you add the terms of the sequence and divide by how many there are. With the sequence in a being three consecutive counting numbers dividing by three it is the same as averaging them Because you have 3 terms and a division of 3. That is why it is always divisible.
0: No Elements of Viable Argument	Includes, but is not limited to, assessments with • No response • An affirmations of the claim • Response that is not relevant/related to the task • Assertion that the original claim is false and no counterexample provided and *does not* included enough detail to reasonable assure the rater that the role of counterexample in mathematical argumentation is understood. • Empirical support, regardless of the number or "size" of examples chosen, with no skepticism expressed (as in example support 1 in Score 1. • A response scores a 0 for any kind of empirical argument, regardless of the number of examples or magnitude or span of the examples, unless the examples are accompanied by an expression of uncertainty about the true of claim based on the examples or some attempt at a general argument. **Example response 1:** [Blank page or completely irrelevant notes or symbols] **Example response 2 [Empirical/not relevant]:** 4+5=9+6=15/3=5. **Example response 3 [Empirical]:** The sum of three consecutive numbers divided by 3 comes out a whole number; for example, 77+78+79=234, 234/3=78. **Example response 3 [False counterexample and no explanation of why the example (or thought experiment) serves as a counterexample to student's interpretation of the task]**.

Problem 2

Maria claims she has two different computation approaches that produce the same answer when using a secret number. Approach 1: Maria takes her secret number, adds 5 to it, and then multiplies the entire result by 3. Approach 2: Marie takes her secret number, multiplies it by 3, and then adds 5.

a) Develop a variable expression for each of Maria's approaches.
b) Set the two expressions equal and solve.
c) Do Maria's two approaches with the secret number produce the same result? Develop a viable argument for or against your response.
d) Describe the type of argument you used in part c). Why do you believe it is viable?

Purpose of the item: Assess whether student acknowledges that a "there does not exist" claim is a generalization and whether the respondent can produce an argument that addresses all cases, in this class eliminates all real numbers as solutions to the equation 3(x+5)=3x+5.

Claim score: A 3 is given for response that expresses awareness that in order for Maria's claim to be false, all x's must be ruled out as solutions of 3(x+5)=3x+5. Responses that offer examples and conclude Maria's claim is false are awarded a 2 because these respondents understand the equating part of the task, but perhaps not the generality. A 1 is awarded to responses that at least acknowledge that results from operations on numbers are compared.

<table>
<tr><td>Rating</td><td>Description</td></tr>
<tr><td>3: Viable Argument</td><td>If response fully meets the purpose of the item, it should be scored a 2 or a 3.

Response needs (a) and EITHER (b) or (c):
(a) Acknowledgement of the generality of the domain (no starting number can make it work, etc.)
(b) A conceptual insight and sufficient detail about its use to conclude that no x can make the equation 3(x+5)=3x+5 true. Conceptual insight can include but is not limited to:
For all x, approach 1 results in a number larger than approach 2 because both expressions have a 3x, but 15 is larger than 5. (Some students may explicitly identify the difference as 10.) Such conceptual insights involve structure.
Implicit or explicit contradiction: For all x, 3x+15=3x+5 implies 15=5.
(c) Reduction of the equation to a form that serves as a prior result. The following forms can be taken as serving as a prior result: x = x, x = a, a = a, or a = b (this is in the Common Core).

An inconsequential algebra error is okay.

Example Claims:</td></tr>
</table>

	• Student claims that no number can be a secret number; • There does not exist a secret number; or • For all natural numbers, there is not a secret numbers. • “Maria is incorrect” or “Maria doesn’t have a secret number” **Example response 1 [Thought experiment]:** No [the expressions can’t be equal for any x], because when the equation is switched, approach 1 will always be bigger. This is because in Approach 2, you multiply her secret number by 3, then add a single 5. In approach 1, you add her secret number to 5 then multiply it by 3, so unlike approach 2, approach 1 adds 10 to the equation. **Example response 2 [Implicit or explicit contradiction argument]:** Claim: 3(n+5)=3n+5 do not create the same result when n is the same. Foundation: Assume 3(n+5)=3n+5. [Equation solving resulting in 15 not equal 5.] By assuming 3(n+5)=3n+5 is true as Maria claims, and simplifying the equation down, 15=5 emerges which is incorrect. Therefore, 3(n+5) not equal 3n+5… I used a contradiction argument. I assumed Maria’s claim was correct by saying 3(n+5)=3n+5, and as I simplified it down I proved otherwise. It’s viable because it eliminates counterexamples by showing 15 not equal 5. **Example response 3 [Arithmetic errors in an otherwise valid approach to the item]:** (x+5)3=(x)3+5 [distribute] implies (x)3+15=3x+5 [subtract 5 from both sides] implies 3x+10=3x implies x+10 not equal x. **Example response 4 [Generic example]:** [Generic examples used in an argument such as the three above].
2: Elements of Viable Argument	Response includes a conceptual insight that is leverageable to a fully viable argument. For instance, the student might not be sufficiently clear that no (real) number x can solve the equation (i.e., can serve as a secret number). A respondent might say, “It didn’t work” or “the approaches are not equal” instead of saying “no x works” or develop a generic example argument that does not refer to specific algebraic structures. Response includes a conceptual insight that is correct and could be used to develop a viable argument/proof that no x solves 3(x+5)=3x+5; however, the response is lacking in sufficient detail to be considered viable. There must be evidence that they went beyond just noting that the two expressions look different or operate differently. An algebra error results in a general viable argument for a general claim, even though that general claim is not correct.

	Example response 1 [Insight but insufficient detail]: No, they don't [produce the same result for the secret number]. This is because, no matter what number you use with either approach, approach 2 will always produce an answer that is 10 less than approach 1's answer. It produces a number 10 smaller because you are multiplying 3 by a smaller number. **Example response 2 [Insight but insufficient detail]:** [3(x+5)=3x+5 reduced to 15 = 5; however, response lacks explicit acknowledgement of the meaning the equation solving activity. This is distinct from responses that acknowledge that the equation 15=5 is not true (or absurd) and concluding from this that no x can serve as the secret number.] **Example response 3 [Insight but insufficient detail]:** [3(x+5)=3x+5 is reduced to 3x+15=3x+5, then "no matter what value you put in, the results will never be the same." Because it is not reduced to a canonical form like 15 = 5 where it can be assumed that a prior result is being appealed to, the response needs more detail about WHY they can never equal (for instance, 10 apart for all x, or reducing to 15 = 5, or the 5 gets multiplied by 3 in one case and by 1 in the other). **Example response 4 [Conceptual insight but lacks sufficient general claim]:** 1. (x+5)3; 2. (x*3)+5. x=1. 1: 18. 2: 8. No, because when she adds the five first, she is also multiplying the five, so it is bigger. Whereas when she multiplies first the five does not get multiplied. [Students appears to acknowledge 3x is in both expressions but the constant terms are different.]
1: Limited Elements of Viable Argument	Here are several ways to get a 1: • Recognize the generality of Maria's claim (and make a non-empirical argument) • Looking for a general argument / conceptual insight for why the two approaches never agree for all x • End up with an algebra error that ends with *equal* expressions and argues that the results are always equal. Here are more details: • An algebra error could result in a "there exists" argument for something like "x = 2." • The student might interpret Maria's claim as a general claim (for all) and gives a viable counterexample argument. (This is not a 2 because it represents a substantive change in the task, from a targeted general claim response to a specific claim response.)

	• The student might say that the two numbers are not equal because the expressions are different (for instance, "3x+15 and 3x+5 are different"), but the equation is reduced no further and they say nothing about why they are always different, or why they are unequal for all x. Student says that the two approaches will never produce the same results, but their reason is because the two procedures are different or that there ends up being formal structural differences between the expressions. For instance: • "The two equations are different." • "You get different things because it's a different order of operations." • "The constant terms are different." • "The output of the expression depends on x." To get above a 1 they need to give a reason why the two results must be different that goes beyond these. **Example response 1 [Differences in constant terms]:** Maria's outcomes are not the same because the numbers that are having 5 added to it are different in each approach. **Example response 2 [Dependency]:** (y+5)X3; (y+3)X5; rely depends on what is y=?, because if they get the same answer [sic]. **Example response 3 [Order of operations]:** No because you have to the correct order of operations… the second… creates a different answer. **Example response 4: [Formal differences between the procedures or expressions]** No, Maria's two approaches don't produce the same result because one of the equations multiplies all of the numbers by 3, and the other one multiplies just the x. So, that makes these two equations totally different.
0: No Elements of Viable Argument	Response includes, but is not limited to, assessments with • No response. • An example (solution to the equation) is presented, which could serve as a false counterexample, but the student does not acknowledge its role as a counterexample to a general claim. ("x = 2 is an answer") • Student only restates claim or only affirms claim. • Lack of conceptual insight. • Empirical support for a general claim (and no evidence that they looked for a general argument or that they recognize the inadequacy of an empirical argument).

	• Response is unclear as to whether or not the student understands that there are two processes (expressions) to equate and examine as to why they do not equate. **Example response 1 [Empirical]:** x=1; 3(1+5)=18; 3X1+5=8. No. I made x=1, and the results were very different. **Example response 2 [Affirmation-appearance]:** They [the approaches] should be equal because they are the same. **Example response 3 [Empirical]:** I pretended x= a number [examples x=6 illustrate]. I found that the 2 outcomes were different. Approach 2's outcome was smaller than approach 1's. I pretended x was 6, 2, and 3. My argument was true.

Problem 3

Jordan calls the numbers 10, 20, 30, 40, 50, 60, 70, 80, and 90 the "tens." Jordan believes she has found a cool method for finding the remainder when she divides any "ten" by 9. Jordan says the remainder is the same as the digit in the tens place of the original number. For example (*division box is shown*), she tested 70 and finds it works because the remainder 7 is the same as the tens digit in 70. It works for the other numbers as well. 10 ÷ 9 has remainder 1. 20 ÷ 9 has remainder 2. 30 ÷ 9 has remainder 3.
Is Jordan correct that her remainder method works for all "tens" less than 100? Develop a viable argument for your response.

Purpose of the item: Assesses whether the respondent understands that generalizations with a finite domains are true if and only if there are no counterexamples in the domain of the claim. Proficient students either 1) find a counterexample and give a complete counterexample argument, 2) exhaust all cases 40-80 (when the claim is true), or 3) provide a viable general argument such as a generic example, thought experiment, or variable-based argument for cases 40-80.

<table>
<tr><td>Rating</td><td>Description</td></tr>
<tr><td>3: Viable Argument</td><td>If response fully meets the purpose of the item, it should be scored a 2 or a 3.

Respondent presents a counterexample of 90 and shows it has the desired properties:
• Does not have a remainder of 9
• Remainder is not the same as the tens digit as asserted in Jordan's theory
Respondent may present extraneous information without lessening this score, such discussing conforming cases.

Example Claims:
• There exists a case in the domain of Jordon's claim that does not have the remainder property (or doesn't work)
• Jordon is not correct.

Example response 1: Jordon is incorrect. Most of her claim is true but this is how I would change it: The remainder method works for all tens less than 90. 90 is a multiple of 9, so therefore 9 goes into 90 equally. The method works for all other 10s (10, 20, 30, 40, 50, 60, 70, and 80), but it does not work at 90.

Example response 2: No, because 90 is equally divisible by 9, which means there isn't a remainder there.</td></tr>
</table>

2: Elements of Viable Argument	Respondent presents a counterexample of 90 but the demonstration that the example has the desired properties is incomplete. OR Respondent presents a general argument that is viable for cases 10 through 80 (meaning based in a conceptual insight and appropriate details linking the conditions and conclusion are present) but student fails to recognize that this otherwise viable approach fails at 90. This could include a misunderstanding of division algorithm (e.g. 9 is a remainder when dividing 90 by 9). **Example response 1** [**Affirms Jordan's claim with CI (conceptual insight) argument relevant for cases 10-80**]: When a number <10 is multiplied by 10, the outcome is that <10 number then has a 0 after it and that digit becomes the 10s place, because you have 10 of that number. When you only have 9 of that number you get to the next "tens" by adding it again to get 10 of that number. Because of this, a tens number will have its digit as the remainder when divided by 9 because it needs one more of the original <10 number to get to the "tens." **Example response 2** [**Affirms Jordan's claim with CI argument relevant for cases 10-80**]**:** She is correct; because, for example, when you multiply 3 by 10 you get 30. When you multiply 3 by nine, you get 3 less than that. Therefore, the remainder will always be the same as the tens digit **Example response 3 [Viable deductive argument for a claim that did not include 90]:** Jordon is correct because 9 times any whole number below 10 is that number times 10 minus itself. For example, 9X5=5X10-5. This equals 45… plus 5 is 50. Thus, 50/9=50-5.
1: Limited Elements of Viable Argument	Respondent presents a general argument that for cases 10 through 80 based ideas sufficient to assure the rater that the students has an appropriate and useful conceptual insight but that insight may be poorly expressed (evidence that they are looking for some appropriate general structure). Also, the student may fail to recognize that this otherwise viable approach fails at 90, and the general argument is not completely viable because some the needed details for using the conceptual insight to link to the claim are missing. Or

<table>
<tr><td></td><td>Respondent expresses awareness of the finite domain of the claim and awareness that all cases must be exhausted but shows no work.

Or

Respondent exhausts all cases 10 through 90 but misinterprets the results or makes a computational error.

OR

Student offers an incorrect counterexample, perhaps misinterpreting part of the task or using an example outside the domain of the claim. (For instance, they misinterpret “remainder” or dividing in a significant way, such as the first digit in the decimal expansion of the result.)

Example response 1 [“Counterexample” outside the domain of the claim.]: 100/9=11R1...the remainder says that her argument is not valid.

Example response 2 [“Counterexamples” based in misinterpretation of remainder]: 90/10=10, 20/9=2.22222, 10/9=1.1111111, 30/9=3.3333. This method does not work because I found four counterexamples and 3 are decimals in the ones place but are decimals and a 10 which is 90/9=10 which is in the tens place which is not in the ones place.

Example response 3 [Conceptual insight poorly expressed]. Yes, Jordon is correct because when you divide a ten’s number by ten, you don’t get any remainder whatsoever. Therefore, when you divide a 10’s number by 9, you get the number as it would be in the one’s place.</td></tr>
<tr><td>0: No Elements of Viable Argument</td><td>No response or a response either:
Affirms Jordon’s claim based on empirical tests for a subset of the claim’s domain.
Asserts Jordon’s claim is false but offers no counterexample.
Reflects a significant misinterpretation of the task.</td></tr>
</table>

	Example response 1 [Affirms Jordon's claim but does not exhaust all cases; perhaps has a content knowledge issues as well]: 40/9=4.4. 50/9=5.5. 60/9=6.6. So it's true but not exact. You have to round so not exactly true. **Example response 2**: I agree with her work because there is no flaw. **Example response 3:** I think some of it is wrong because in the examples 10/9, 20/9, 30/9, she said they all -3 but it's incorrect because they are all in decimal forms so she is wrong.

Problem 4

A student in your class makes the following claim: For all perfect squares between 0 and 50, none of these numbers have a remainder of 2 when divided by 4. Develop a viable argument for or against this student's claim.

Purpose of the item: Assesses whether the respondent understands that generalizations with a finite domain are true if and only if there are no counterexamples in the domain of the claim. Proficient students either 1) find a counterexample and give a complete counterexample argument (an incorrect response for Problem 4), 2) exhaust all cases between 0 and 50 (a correct response for this task), or 3) provide a viable general argument such as a generic example, thought experiment, or variable-based argument (can be a correct response for this task).

Rating	Description
3: Viable Argument	If response fully meets the purpose of the item, it should be scored a 2 or a 3. Respondent presents: • An exhaustive argument, perhaps overlooking 1^2, and affirms the original claim; the needed details to show that both the conditions and conclusions are satisfied for all cases are present. • The conceptual insight that all odd perfect squares must have a remainder of 1 or 4 and even perfect squares have remainder 0; argument has sufficient details for why the insight is true (e.g., an odd number squared is odd; and even number squared must have a 4 (pair of 2's) in its prime decomposition). A minor computation error is possible. **Example Claims:** • Student claims that no perfect squares between 0 and 50 have remainder 2 when divided by 4 • Student claims "yes," "the claim is correct," or some other affirmation of the original claim. **Example response 1 [Detailed Viable exhaustive argument]: "**For all perfect squares between 0 and 50, none have a remainder of 2 when divided by 4. [Student includes a table with cases 4 thru 49 (2*2=4 thru 7*7=49) in column 1 and corresponding division algorithm and the results in column 2.] By laying out all the perfect squares up to 50 (7*7=49) and showing the quotient for each when divided by 4, I've proven that no perfect square between 0 and 50 has a remainder of 2 when divided by 4."

	Example response 2 [Sufficiently detailed Viable exhaustive argument]: "This claim is true. 4, remainder 0; 9, remainder 1; 16, remainder 0; 25, remainder 1; 36, remainder 0; 49, remainder 1. There are no counterexamples to the claim." **Example response 3**: Student gives a general argument that classifies all remainders as either 0 or odd by considering **cases of perfect squares**. Student argues that odd perfect squares have a remainder 1 or 3 when divided by four, and that even perfect squares have a remainder of 0 when divided by four and gives structural response for both of these facts (e.g., because even perfect squares result only from squaring an even number and the two multiplies to a four). This conceptual insight may be expressed in prose, diagrams, generic examples, or variable arguments. **Example response 4**: Student gives a general argument that classifies all remainders as either 0 or 1 by considering **cases of even or odd numbers and squaring them**. Student explains why the squares of evens and odds cannot have remainder of 2 through cases: "even numbers are squared to get a multiple of 4 and odd numbers squared are odd." In particular, why even times even is four is addressed. This conceptual insight may be expressed in prose, diagrams, generic examples, or variable arguments.
2: Elements of Viable Argument	Respondent presents: • An exhaustive argument, perhaps overlooking 1^2, and affirms the original claim. Shows awareness that all cases must be tested, but perhaps shows only a subset of the cases. Some of the needed details to show that both the conditions and conclusions are satisfied for all cases are absent; however, there are enough details about the context to assume the student understands the task and the cases to exhaust. For instance, the student has an exhaustion argument but has incorrect or unclear meaning of "remainder." • The conceptual insight that all odd perfect squares must have a remainder of 1 or 4 and even perfect squares have remainder 0; however, the argument has some but not sufficient details for why the insight is true. For example the argument states that odd times and odd is odd and even times and even is a multiple of 4 but does not explain why the "multiple of 4 conclusion is a necessity".

	Example response 1 [Exhaustive argument with limited details showing the conclusion is met for all cases]: 2^2=4; 3^2=9; 4^2=16; 5^2=25; 6^2=36; 7^2=49. I think he is right, because after analyzing all perfect squares, I notice they all followed his rule. **Example response 2 [Exhaustive argument with unclear details showing the conclusion is met for all cases]:** ¼=.25; 4/4=1; 9/4=2.25... 36/4=9; 49/4=12/25. None of these remainders are 2. **Example response 3 [List of cases and says tested all cases, but presents empirical argument]:** The student is correct. Any perfect square under 50 will not have a remainder of 2 when divided by 4. I tested all perfect squares and in all outcomes, the student is right. [Work:] Perfect Squares: 1, 4, 9, 16, 25, 36, 49. 49/4=12R1. True!
1: Limited Elements of Viable Argument	Respondent presents: • Incorrect counterexample due to incorrect division algorithm or incorrect interpretation of the results of the division algorithm. • Incorrect counterexample due to notion of perfect square. • Exhaustive argument or claim of exhaustion but the details provided are insufficient to convince the rater that the students understands the context of the task well (e.g., perfect square, remainder, or divisor are not clearly articulated). • Respondent expresses awareness of the finite domain of the claim *and* awareness that all cases must be exhausted *but* shows no work. **Example response 1 [Incomplete knowledge of division algorithm: quotients]:** The claim is incorrect because there is a remainder of 2 when you divide 1 by 4. [Student work displays 1/4=0.2 R 2. **Example response 2 [Incorrect understanding of the conditions]:** This student's claim wrong. 50/4=12 R2. Because when 50 is divided by four and 4 fits into 50 12 times. And 50-48 is two then 50 divided by 4 is remainder 2. **Example response 3 [Incomplete knowledge of division algorithm: remainder]:** I disagree because 25/4=6.25 or 6 1/4.

	Example response 2 [Exhaustive argument with significant misinterpretation]: There exists no counter example. Any possible C.E.=sqrt(x)/4=2. ¼=.25 not equal 2. 2/4=.5 not equal 2. 3/4=.75 not equal 2. 4/4=1 not equal 2. 5/4=1.25 not equal 2. 6/4=1.5 not equal 2. 7/4=1/75 not equal 2. For there to be a counterexample there has to be a number from 1 to 7 that is divisible by 4 and =2. None exists.
0: No Elements of Viable Argument	Respondent presents: • No response, • Affirms claim to have tested "almost." • Affirmation, restatement, or rejection of the original claim without support, • No counterexample presented, and no clear references to objects in the conditions or the conclusion. • A response expressing a significantly incomplete or incorrect interpretation of the task with no argument or expresses a correct interpretation that has no elements of viable argument • [Empirical support] A response expressing no knowledge that each case must be examined or a general argument must be constructed. **Example response 1:** [Incomplete interpretation] I do not particularly understand squaring; however I'm sure there is a rule that would prove this. **Example response 2:** [Affirmation with little to no support] This true, because they all either have a remainder of 1, or 3. **Example response 3: [Affirmation and claims to have tested "almost" all].** This student is correct because I attempted almost all perfect squares none had a remainder of 2.

Problem 5

Wuan is trying to prove whether the following statement is true or false: When you add any two odd counting numbers, your answer is an even counting number. Wuan's argument: Odd numbers are numbers that come in pairs with one left over. For example, 7 and 9 are odd numbers because they can be written as 7=2×3+1 and 9=2×4+1. Because I can add the leftover ones and factor out a 2 from each term, the answer is 2 times another integer. Because the sum can be expressed as a double, the sum is even.

a) Is Wuan's argument viable? Explain. Be specific. Does Wuan's argument work for all odd numbers? Explain. Be specific.
b) Wuan wishes to create a variable based argument for the same claim. Show how to use Wuan's argument using variables to represent any two odd numbers.

Purpose of the item: This item assesses student's ability to recognize a viable generic example argument and to articulate reasons why this generic example argument is viable. In particular, proficient students recognize that the particular example shows the mathematical structure, shared by all cases, that links the conditions to the claim. This structure is that all odds can be written as a double plus 1 (condition) and when summed (condition) the doubles pair to form a double and the 1's pair to form another double so that the overall sum is a double.

*****Note: Award the highest score supported by features of a particular score found anywhere in the response, either a) or b), regardless of the prompt. Disregard information that might lessen than score. We do not penalize students for constructing a viable argument then developing more, less viable support.*

Reading Score: To award a 2 on the "read correctly" score, the rater must have evidence that the student is critiquing Wuan's argument, not his just claim. A student who writes, "Wuan is correct the sum of two odd numbers is an even number" and only other sentences that indicate they are addressing the truth of his claim receives a 1.

Rating	Description
3: Viable Argument	If response fully meets the purpose of the item, it should be scored a 2 or a 3. Response includes all of the following features: • Explicitly mentions that pairs plus one structure in Wuan's applies to all odd number and is a defining trait of "odd." It is not necessary that the word "definition" or a formal definition is mentioned. • Explicitly mentions or illustrates algebraically or with an appropriate diagram that Wuan demonstrates that the ones pair to construct a sum of even numbers, 2 or 3 even numbers, which is even.

	Response must be explicit about Wuan's example use: that his structure and logic apply to all sums of odds because the properties of objects in the conditions are linked to the conclusion in a manner that makes the conclusion a necessity. Response may notate the two odd numbers as being the same, but does not use this specific property in the argument. For instance, [2n+1 + 2n+1 = 2n + 2n + 2, which is sum of evens], or [(2·3 + 1) + (2·3 + 1) = (2·3 + 2·3 + 2), which is sum of evens]. Response has most of the features described above, but not all, and is able to develop a completely viable argument variable based argument akin to Wuan's generic example computation. **Example Claims:** • All sums of odd numbers are even **OR** • If odd plus odd, then even **OR** • The sum of two odds is even **OR** • Wuan is correct. **Example response 1:** Part a.1). "Yes, it only uses one example, but it explains in the narrative link. a.2) Yes, all odd numbers are two times another number, plus 1." Part b) "When you add any two odd numbers, you get an even. (2n+1)+(2x+1)=2n+2x+2=2(n+x+1). You can factor out 2, so therefore it is even." **Example response 2 [Same two numbers expressed as referent, but referenced generally]:** Yes, because as Wuan explained, an odd number is just an even number, +1, and since even numbers can be written as 2x, an odd number plus another odd would be 2x+2x+2, therefore making the result even. **Example response 3: [Generic example diagram]** "Both numbers have a left over 1. When you combine the one left over to the other one, it makes it even. That's why it works all the time." [with the following diagram, which illustrates the pairing process for 7 + 9] 
2: Elements of Viable Argument	Responses that:

	• Are akin to those described in a score of 3; yet, some important feature or features of the argument is (are) missing or is (are) too vague to be considered viable. Examples include but are not limited to: Respondent notes that all odd numbers share the odd structure Wuan describes and that the 1's sum to an even, but the respondent does not explicitly address the divisibility by 2 component of Wuan's argument, or note that the evens sum to evens. • Present a viable argument but it is unclear whether or how it responds to Wuan's argument. For example, presents an argument about the ending digits of odd numbers (cases). • Present an argument intended as general but in truth addresses a general proper subset of the pairs of odds, or presents an argument that is not quite fully general. For instance, an argument that treats only cases where the odd numbers are the same, either with algebraic notation [2n+1 + 2n+1 = 2(2n+1) rather than 2n+1 + 2m+1 = 2(n+m+1)] or as a generic example [(2·3 + 1) + (2·3 + 1) = (2·2·3 + 2)]. • Assert that Wuan's argument does not apply to all cases because it is only one example of sums of two odds, and yet, respondent is able to develop a viable variable-based argument akin to Wuan's in part b). In other words, the respondent recognizes the generalizable structure in Wuan's argument but is not willing to call example-based arguments viable. • Apply or replicate Wuan's process with variables, suggesting that the generality of the process presenting in Wuan's example is followed as general, but the use of variables does not adequately convey what the variables represent. **Example response 1 [Nearly all the pertinent structure] :** "Yes, Wuan's argument would work for all odd numbers. Let's pick 9. The number right before 9 is 8. 8 comes in the pair 4-2. All even numbers can be divided by 2, and all odd numbers come after an even (except 1). So you can take the pair that the previous number, an even had, and simply add 1… N+n=(2x+1)+(2n+1). Because there are 2 1s, 1 goes to each "side" and when you multiply by 2 the number becomes even. Therefore his logic is correct." **Example response 2 [Nearly all the pertinent structure]:** "Yes, odd numbers are just even numbers +1, so when you add them together, the +1s become a 2, and it is even."

	Example response 3 [Example-based argument not viable, produces the variable-based equivalent]: Part a.2). "No [Wuan's argument does not work for all odd numbers], because his argument consists of one example." Part b). "Let x and z be odd, x=2t+1 and x=2E+1. Z+X=(2t+1)+(2E+1)=2(T+E++(1+1)=2t+2E+2=2(t+E+2)." **Example response 4 [Same two numbers expressed as referent and referent appealed to in manner that questions generality]:** (2n+1)+(2n+1)=4n+2=2(n+1). Because a 2 can be factored, the answer will be divisible by 2. *[Here, generality is lost in the second equals sign, since the result "even" through factoring is dependent on the two odds being equal.]* **Example response 5 [Presents alternative viable argument; cases]:** Yes, his examples supports his claim. Yes, all odd numbers between 1-10 added = an even number, and an odd number is always at the end of an equation, the number one of the ones between 1-10. 71+73=14<u>4</u>. 65+67=11<u>2 [underlining as in the original.]</u>.
1: Limited Elements of Viable Argument	Responses that • Include explicit, yet vague, <u>attention to at least one of the pertinent</u>, general structural features in Wuan's argument, a feature that can be applied to other odd numbers. For example, the respondent might mention that 1's in the expressions of "odd" pair to become an even. • Assert that <u>Wuan's argument does not apply to all c</u>ases because it is only one example of sums of two odds, and respondent is <u>unable to or simply does not develop a viable variable-based argument</u> analogous to Wuan's in part b). • Express skepticism of generalizations and their support by providing an incorrect counterexample. • Apply Wuan's process to other odd numbers, suggesting that the generality of the process presenting in Wuan's example is followed as general, but the response is too vague about "all cases" to warrant a higher score. • Engages critically in Wuan's argument, perhaps critiquing one of the steps as not applying to all cases and perhaps offers a "local counterexample."

- Presents converse reasoning, such as arguing that every even slips into odd summands.

Student is awarded at least a 1 if these features are found anywhere in the response, a) or b), regardless of the prompt.

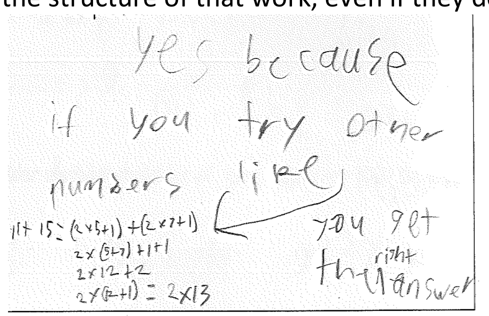
Example response 1 [1's pair, vague]: "Yes, because all of the extras from the odd numbers fill the gaps."

Example response 2 [1's pair, vague manner, not analogous to Wuan's approach]: "Yes, Wuan is viable. Subtract 1 from every odd number. (These 1's =2.) The odd are now even. Add the evens together (don't forget the 2!) and your answer is complete."

Example response 3 [Incorrect counterexample]: "Claim: There exists Wuan's claim is false [sic]. Foundation: 1+9=(1X1)+(2X4+1) [Several steps of unclear computation on the right hand side.] Narrative link: Because of 1+9=27 is not [unclear] even… 1+9=10."

Example response 4 [Reference to structure, extremely vague]: Part a.2) "True [that Wuan's argument works for all odd numbers] because I found that for all odd numbers they all equal something with one left over."

Example response 5 [Potentially empirical argument, but with the beginning of understanding structure]: [The student understands the way Wuan's work could be adapted for other cases, meaning they understand the structure of that work, even if they don't understand the argument.]


Yes because if you try other numbers like
11+15=(2×5+1)+(2×7+1)
2×(5+7)+1+1
2×12+2
2×(12+1)=2×13
you get the right answer

	Example response 6 [Engaging in the argument, incorrect critique; local counterexamples, yet addresses argument critically]: I agree that they do all add to an even but when you use three and one there isn't that two so the way he explains it isn't always true. No because in the 1 and 3 there is not two's. [Student is apparently critiquing the first step in the process, writing each odd as 2k+1, and providing a "local counterexample."]
0: No Elements of Viable Argument	Responses with • <u>No Response</u> • <u>Affirmation with empirical support</u>. • <u>Affirmation</u> with details that are <u>too vague to assess</u> whether respondent is critiquing Wuan's argument. • Conceptual insight(s) that <u>does not appear to be relevant</u> or does not appear to have potential to leverage toward a proof. • <u>Empirical support</u> for Wuan's claim without critiquing Wuan's approach • Supports Wuan's claim with an argument that is distinct from (<u>not analogous to) Wuan's approach</u>, without critiquing Wuan's approach, even if the alternative support is otherwise viable. **Example response 1 [Empirical support]:** "Yes because 5+5 or 7+3 will equal an even number." **Example response 2 [Vague]:** "It is viable because his reasoning is sound. If you follow what he does you will get to the right answer. You could do it in a simpler fashion and you would come to the same conclusion in less time." **Example response 3 [Conceptual insight with no apparent path toward proof]:** "Yes, because every odd number you add or times by 2 you get doubled and that makes it legitimate." **Example response 4 [Does not appear relevant]:** a1). Part a1). "I do not understand Wuan's claim. But I think it is viable. Part a2). Yes, any off counting number equals an even. Part b) I don't know how to answer this question, x+y=c, x=odd, y=odd, c=even. **Example response 5 [Empirical without attending to Wuan's approach]:** "Yes, because I know that it will work, 1+1=2, 21+25=46. So it works with all of them."