BENEFITS OF ANALYZING CONTRASTING INTEGER PROBLEMS: THE CASE OF FOUR SECOND GRADERS

Mahtob Aqazade
Purdue University
maqazade@purdue.edu

Laura Bofferding
Purdue University
lbofferd@purdue.edu

Sherri Farmer
Purdue University
farmer10@purdue.edu

In this study, we explore four, second graders' performances on integer addition problems before and after analyzing contrasting cases involving integers. The students, as part of a larger study, participated in a pretest, small group sessions, one short whole-class lesson on integer addition, and a posttest. Based on their integer mental models and scores on arithmetic and transfer problems, each student progressed, although in different ways. We use these instances and their interactions in their group sessions to describe their progressions.

Keywords: Number Concepts and Operations, Elementary School Education, Instructional Activities and Practices

Students start learning about negative numbers and interpreting integers in real-world situations (e.g. temperature, account balance) in sixth or seventh grade; yet, they learn whole number concepts starting in kindergarten (National Governor's Association Center for Best Practices & Council of Chief State School Officers, 2010). During this large time gap, students solidify ideas about numbers and addition based on whole number experiences, and when they encounter negative numbers, they try to make sense of them using this prior knowledge in various ways (e.g. treating negative numbers as positive numbers or zero) (Bofferding, 2014). Helping students bridge the gap between whole numbers and integers by providing them with helpful instructional experiences is important.

One possible way to bridge the gap is to shorten it by introducing negative integers earlier. There is recent evidence that young children are capable of learning about and using negative numbers even in the first grade (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011; Bofferding, 2014; Behrend & Mohs, 2005/2006). Another possible way is to help students make connections between the two types of numbers. Findings by Gentner (2005) suggest there is evidence that learners, who have opportunities to compare two analogous cases, are more likely to succeed at a "relational transfer" than those who have not compared the cases (p. 252). However, prior investigations in students' learning from contrasting cases have mostly focused on older students (e.g. Rittle-Johnson & Star, 2011).

In a larger study, we explored the potential of having second graders analyze different contrasting cases to see if it could help them learn about negative integer addition. Through case studies of two pairs of students, we aim to address the following research questions:

1. How do four, second graders' integer mental models and performance on integer addition problems change after analyzing contrasting cases of integer addition and participating in one lesson on the topic?
2. In attempting to account for these changes, what will the students notice and describe during the analysis of the contrasting cases?

Theoretical Framework

Integer Mental Models

Bofferding (2014) categorized first graders' order and value mental models into initial, synthetic, and formal, along with two transitional levels, which exhibit parts of both. Students with initial integer mental models ignore negative signs, operate with negative numbers as if they were positive

Wood, M. B., Turner, E. E., Civil, M., & Eli, J. A. (Eds.). (2016). *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Pyschology of Mathematics Education*. Tucson, AZ: The University of Arizona.

numbers, or order them correctly but treat them as having positive values (Bofferding, 2014). As they transition away from an initial mental model, some students give new meaning to the negative sign and treat it as a minus (transition I mental model). Therefore, they sometimes treat negative numbers as zero (as if the number is subtracted from itself) and sometimes treat them as positive numbers (numbers that have not been taken away yet). This reflects a focus on the binary (or subtraction) meaning of the minus sign (Bofferding, 2014; Vlassis, 2004).

Students with synthetic integer mental models acknowledge that negative numbers are less than zero but consider larger negatives as "more" (Bofferding, 2014). This focus on absolute value may lead them to reverse the order of negatives. Before developing a formal mental model for integers, some students will both consider negative numbers with larger absolute values as greater than smaller ones and treat them as less, depending on the tasks (transition II mental model) (Bofferding, 2014). As students move toward having a formal mental model, they develop a better understanding of the unary meaning of the minus sign (Bofferding, 2014; Vlassis, 2004).

Contrasting Cases

The key differences among the integer schemas described above are the meanings given to the symbols involved (i.e., the negative sign) and the relations established between order and value. One method for students to notice important relations in problems is by analyzing contrasting cases (e.g., Rittle-Johnson & Star, 2011; 2009, 2007). For example, students who analyzed contrasting alternative solutions in algebraic equations showed better improvement in their procedural knowledge than students who analyzed solution methods presented sequentially (Rittle-Johnson & Star, 2007, 2009). In one case focused on algebraic manipulations, three different comparison groups compared equivalent problems, different solution methods, or different types of problems. Students who compared solution methods gained greater procedural flexibility and conceptual knowledge than students in the other two groups (Star & Rittle-Johnson, 2009). Durkin and Rittle-Johnson (2012) also found that providing students with opportunities to compare incorrect and correct answers can help students improve their understanding of new concepts. Further, others found that even students with low performance on a pretest who later compared correct and incorrect examples of equations improved more in conceptual knowledge on a posttest (Lange, Booth, & Newton, 2014). In this study, we investigate second graders' understanding of integer addition before and after they analyze contrasting cases on the topic.

Methods

Participants & Setting

The larger study, from which this data is drawn, took place at two public schools in a rural mid-western district where about 32.2% of students are English-language learners and 75.2% qualify for free or reduced-lunch. Students were recruited from the second grade population, and 109 of the original 110 participants completed all aspects of the study. After a pretest, students were assigned randomly to groups that analyzed different types of contrasts.

For this study, we chose 4 students to investigate more closely. Because students in the larger study made sizeable gains from pretest to posttest, we chose the two initial case study students to exemplify this change. First, we narrowed the pool to students who had gained at least 30% from pretest to posttest. This resulted in 9 students out of 36 from the intuitive contrast group, 13 students out of 35 students in the conflicting contrast group, and 6 students out of 36 students in the same type, sequential group. We chose to look at students in the intuitive and conflicting contrast groups since they had made slightly more gains than those in the sequential group. We then selected one student from each of the remaining two groups who had 60% or above correct on the session questions and had a consistent partner across the small group sessions. We also included their consistent partners, who had similar pretest scores.

Design and Materials

Pretest. The pretest included three ordering questions (two involved filling in missing numbers on a number path and number line; we ignored the third because students had trouble interpreting it). Students also solved sixteen temperature comparison questions (eight where they circled the hottest temperature and eight where they circled the coldest) that targeted their understanding of integer values and four directed magnitude questions. For the arithmetic questions, they solved four positive integer addition and subtraction questions and then fourteen integer addition questions. Finally, they solved two types of transfer problems: three integer addition problems with three addends and six missing-addend addition problems. Overall, the highest possible score for the pretest was 50.

Small group sessions. After students were randomly assigned to contrast group, they participated in 2 small group sessions. During the small group sessions students analyzed sets of contrasting integer addition problems with accompanying illustrations (both solved correctly and incorrectly) and talked about what was similar and different in the problems and pictures. They also talked about why some of the answers were wrong and how they could use the pictures to solve the problems. The illustration contexts involved (a) a gingerbread man moving on a number path, (b) ants going above and below ground next to a vertical number line, (c) positive and negative chips, and (d) a folding number line (see Tsang, Blair, Bofferding, & Schwartz, 2015). For some of the contrasts, students had to write down their thoughts without discussion, and on others, students talked about their answers aloud. The contrasts the two groups analyzed are shown in Table 1. Each intuitive contrast and conflicting contrast group session took about 20 minutes. At the end of each session, students solved 6 related integer addition problems for a total of 12 problems by the end of both sessions.

Table 1: Contrasts that the Students Explored in their Groups

Illustration Context	Intuitive Contrast, Session 1	Conflicting Contrast, Session 1	Intuitive Contrast, Session 2	Conflicting Contrast, Session 2
Gingerbread	3+5 vs. -3+-5	3+5 vs. 3+-5	-2+5 vs. 2+-5	-2+-5 vs. -2+5
Ants	2+7 vs. -2+-7	2+7 vs. 2+-7	-3+6 vs. 3+-6	-3+-6 vs. -3+6
Chips	6+4 vs. -6+-4	6+4 vs. 6+ -4	-5+5 vs. 5+-5	-5+-5 vs. -5+5
Folding number lines	5+3 vs. -5+-3	5+3 vs. 5+-3	-7+2 vs. 7+-2	-7+-2 vs.-7+2

Whole-class instruction. After all students completed the two sessions, they participated in a 30-minute, whole-class lesson on negative numbers and addition problems with integers. Students learned that adding a positive number to an integer corresponds to moving the gingerbread boy up on the number path, and adding a negative number corresponds to going down on the number path. For example, to solve -2 + 2, the gingerbread boy first moves down 2 (more in the negative direction) and then up 2 (more in the positive direction), ending up at 0, where he started. This led to the introduction of the term “zero pair.” Students explored a few more zero pairs on the number path and then played a card game based on zero pairs with a partner. Students had a stack of -1 cards and a stack of +1 cards (much like with positive and negative chips) and a dice with negative and positive numbers on it to play with. Students rolled the dice and collected the appropriate cards. Using the cards in their hand, their goal was to make zero pairs and be the first person to run out of cards.

Post-test. The posttest contained the same items as the pretest with a couple of additions. We added 3 questions asking students to determine which integer is closest to 10 and 3 more addition problems with 3-addend numbers where it was advantageous to make zero pairs. The highest possible score on the posttest was 56.

Analysis

To analyze the data, we first classified each students' integer mental models on the pretest and posttest based on Bofferding's (2014) framework. We investigated students' answers on the order and value questions and their use of the minus sign on both pretest and posttest. We also referred to explanations they provided in the groups sessions to clarify their mental models.

Then, we transcribed the group sessions for the case study students to identify their contributions to the discussions and identify how each student interpreted negative numbers in the various contrasts and which meaning of the minus sign they used for their interpretations. Finally, we calculated students' gains on the ordering, arithmetic, and transfer questions across the testing situations. We looked across each student's data from the pretest, sessions, and posttest to build a case of their understanding.

Results

Overview

The larger study analysis shows that there were no significant differences between groups (Bofferding, Farmer, Aqazade, & Dickman, 2016); similarly, in our four sampled cases, the students' gains from pretest to posttest were the same regardless of their group. In the intuitive contrast group, X04 started with 22% correct on the pretest overall, which increased to 78% correct on the posttest. On the other hand, W05, her partner, started with 12% correct on the pretest and only increased to 30% correct on the posttest. Students' gains from pretest to posttest in the conflicted contrast group were similar to those in the intuitive contrast group. In the conflicted contrast group, Z08 started with 40% correct on the pretest and went up to 98% correct on the posttest. However, Z10, his partner, started with 30% correct on the pretest and only went improved to 46% correct on the posttest.

Pretest Mental Models

To better distinguish students' performance, we determined their mental models for the order and value questions and then calculated their percent correct for the 14 integer addition problems (with two addends) (see Table 2). Each group had comparable students who made similar high gains in percent correct over the entire set of questions (X04 gained 56% and Z08 gained 58%) or low gains (W05 gained 18% and Z10 gained 16%).

Table 2: Case Study Students' Integer Mental Models and Arithmetic Scores (Pre & Post)

ID	Group	Gender	Mental models Shift	Pretest	Posttest
X04	Intuitive contrast	F	T1-T2 (shift of 2)	7%	100%
W05	Intuitive contrast	F	I-I (shift of 0, but progress)	0%	29%
Z08	Conflicted contrast	M	S-F (shift of 2)	29%	100%
Z10	Conflicted contrast	F	I-I (shift of 0, but progress)	36%	64%

I = initial mental model; T1 = transition I mental model; S = synthetic mental model; T2 = transition II mental model; F = formal mental mode (see Bofferding, 2014)

W05 and Z10. In both groups, one of the pair started with an initial mental model. Both W05 and Z10 filled in only whole numbers on their empty number path and number line. Further, they determined which temperature was hottest and coldest based on absolute value. The main difference was that W05 answered "none" for some of the comparisons. Although they both started with initial mental models, Z10 got 36% of the arithmetic problems correct on the pretest. Looking at the problems she got correct, Z10 was able to solve the problems when she could use the negative sign as a subtraction sign. Therefore, she got -1 + 8 correct (can be solved as 8-1), 7 + -3, 5 + -2, 9 + -9, and -4 + 6. W05, on the other hand, answered all of the problems by ignoring the negative signs (e.g., 9 + -9 = 18, -6 + -4 = 10).

X04 and Z08. Both groups also had a student who demonstrated some use of the negative signs on the pretest. X04 filled in positive numbers on both sides of zero when filling in the number path (i.e., 11, 10, 9, 8, 7, 6, 5 4, 3, 2, 1, 0, 1, 2, 3, 4, 5). Further, she answered the temperature comparison questions by focusing on absolute value. All of these responses would suggest that she has an initial integer mental model. However, she frequently treated negative numbers as worth zero on the arithmetic problems. For example, she answered -5 + 5 = 5 and 0 + -9 = 0. Explaining how she solved -9 + 2, she said, "Minus nine is zero then plus two, so there would only be two left." Her treatment of the negative sign as attached the particular number and indicating that the number was worth zero, suggested that she had a Transition I integer mental model. Finally, Z08 had some knowledge of negatives on the pretest. He filled in negative numbers on his number path and number line, but also included -0 (e.g., -2, -1, -0, 0, 1, 2). When completing the temperature comparisons, he correctly determined that positive temperatures would be hotter than negative ones; however, when choosing which negative temperature was hotter, he chose the larger negative. Therefore, he demonstrated a synthetic integer mental model on the pretest.

Students' Performance in Small Group Sessions

W05 (initial mental model) and X04 (transition I mental model). The intuitive contrast group investigated addition problems with two positive versus two negative numbers in the first session and then compared adding both a positive to a negative number and a negative to a positive in the second session. Based on their conversation, X04 and W05 both noticed the negative sign across the sessions; however, they used the binary meaning of the minus sign or called it line. When asked about differences between two problems (5+3 = 8 vs.-5+-3 = -8) and their corresponding pictures in a folding number line context, X04 explained, "That one [is] normal (referring to the initial picture) but that one has a one and a minus (referring to the second picture)." W05 later continued, "These all there have a minus." In general, W05 primarily focused on the minus sign and made more generic comments, such as indicating that the two problems were not the same rather than pointing to what exactly was not the same. On the other hand, X04 frequently talked about more details beyond just the minus signs. She was more likely to talk about all elements in the problems and to come up with reasons why the information in the pictures matched the problems. On the end-of-session questions, W05 got 6 out of 12 (50%) of the questions correct, and X04 got 12 out of 12 (100%) correct.

Z10 (initial mental model) and Z08 (synthetic mental model). The conflicted contrast group investigated addition problems with two positive numbers versus a positive number plus a negative number in the first session and then compared adding a negative number to a negative versus adding a positive number to a negative number in the second session. Their conversations in the sessions showed that they mostly used the unary meaning of the minus sign and sometimes the binary meaning of minus sign when interpreting negatives. Unsurprisingly, Z08 was more dominant in reasoning about negative integers compared to Z10 and sometimes helped Z08. This occurred when they compared -2 + -5 (where a gingerbread man started at 0, hopped to -2 then hopped -5 more to -7) versus -2 + 5 (where a gingerbread man started at 0, hopped to -2, then hopped 5 to 3). Z10 looked at the ending point on the pictures and said, "This one starts at -7, and this one starts at normal 3." Z08 clarified, "That one actually starts at 0." Z10 then saw that the answers to the problems corresponded to the ending points for the gingerbread man. Z10 tended to focus on the minus as meaning subtraction. When thinking about how to solve -5+5=0 with the positive and negative chips, Z10 explained, "This takes this one (referring to a circled group of -1 and 1) and this one takes this one; these are zeros." On the other hand, Z08 correctly identified negative signs and used them to interpret the ant's movement on the number line when explaining why the ant would go down for 2 + -7: "Cause of negative seven." On the end-of-session questions, Z10 got 3 out of 12 (25%) of the questions correct, and Z08 got 11 out of 12 (92%) correct.

Posttest Mental Models

W05 and Z10. On the posttest, W05 and Z10 from the two different groups still had initial mental models, but they had advanced from whole number to absolute value mental models. Both filled in the number path completely. However, they continued to choose hottest and coldest temperatures by their absolute value. In W05's first session she compared adding two positives versus adding two negatives, and she paid attention to the negative signs. On the posttest, instead of providing only positive answers, she answered all problems by adding the absolute values and making the answers negative. By overgeneralizing the rule that negative problems have negative answers, she only correctly solved arithmetic questions where this strategy led to a correct answer (see Table 3). She was also able to correctly answer -2 + __ = -8 from the transfer problems.

Z10's original session focused on adding two positives and adding a negative to a positive, and she continued to treat negative signs as subtraction signs. Therefore, she performed better on solving arithmetic and transfer problems with 3-addend numbers compared to W05. On the pretest, she solved 8 + -3 + 4 as 8 – (3+4) = 1; however, she correctly answered it on the posttest.

Table 3: Case Study Students' Performance on Arithmetic Problems by Strategies

	Add Absolute Value and Make Negative			Use Negative as Subtraction			Count from Negative		
Test Items	-2 + 0; 0 + -9; -6 + -4; -1 + -7			-4 + 6; -5 + 5; -1 + 8; 5 + -2; 7 + -3; 9 + -9			-3 + 1; -9 + 2; 6 + -8; 4 + -5		
	Pre (n=4)	**Mid (n=3)**	**Post (n=4)**	**Pre (n=6)**	**Mid (n=4)**	**Post (n=6)**	**Pre (n=4)**	**Mid (n=2)**	**Post (n=4)**
X04	0%	100%	100%	17%	100%	100%	0%	100%	100%
W05	0%	100%	100%	0%	0%	0%	0%	0%	0%
Z08	100%	100%	100%	0%	100%	100%	0%	50%	100%
Z10	0%	0%	25%	83%	0%	100%	0%	0%	50%

X04 and Z08. Both X04 and Z08 shifted two mental model levels up after posttest. Both correctly filled in the number path and could answer all "coldest" comparisons correctly. When asked to select the hottest temperature with only negative number choices (e.g. -6, -2, -3), X04 selected, "none". Therefore, she demonstrated a transition II integer mental model; whereas, Z08, who got all of the comparisons correct demonstrated a formal mental model. Overall, they both were able to build on their understanding of negatives after analyzing the contrasts and solved almost all of the arithmetic and transfer problems correctly (see Table 4) on the posttest.

Table 4: Case Study Students' Performance in Transfer Problems (Pre & Post)

	Addition with 3-addend numbers		**Missing numbers**	
ID	**Pre (n=3)**	**Post (n=6)**	**Pre (n=6)**	**Post (n=6)**
X04	33%	100%	17%	83%
W05	0%	0%	0%	17%
Z08	0%	100%	17%	100%
Z10	0%	100%	0%	17%

Conclusions and Further Study

Contrasting cases helped the students focus on relevant details and develop their understanding of negative numbers to varying degrees. By chance, the students with lower scores had been paired with higher performing peers. This pairing throughout the sessions allowed them to share information with their partner. However, they did not seem to benefit equally from the sessions. In line with their lower background knowledge, W05 and Z10 focused on one main idea from their sessions. W05 attended to using negative signs, and she provided negative answers for almost every integer problem on the posttest (instead of only providing positive answers). Her initial session focused on negative problems only having negative answers, which may have contributed to this tendency. Z10 continued to use the negative sign as subtraction. Interestingly, her initial session reinforced the idea that adding a negative is similar to subtraction. Therefore, she might have made more progress if she had made a different comparison first that made her question her interpretation of the minus sign.

Both X04 and Z08 demonstrated that they treated negative numbers different from positive numbers on the pretest, although Z08 was further along in his understanding. They both benefited greatly from their sessions, reasoning in more depth about the problems they were analyzing. Based on these cases, it is not clear if they would have had similar gains if they had been in the opposite group. One of our future goals is to develop cases of all students to determine if there are any trends in students who benefited more from their group versus those who made less progress. This could help teachers determine optimal contrasts for students.

The two students in the conflicting cases group made some gains from the end-of-session questions to the posttest. In the case of Z10, his percent correct on the problems that would be correct if you used the negative as subtraction (e.g., 7 + -3) went down after the sessions and then went back up by the posttest. Z08, made steady progress on the problems that are best solved by counting from negative. These results suggest that especially for that contrast group, the instruction helped them make sense of the problems they had analyzed. Further exploration of the larger dataset will clarify the extent to which this was true for the second graders overall.

Acknowledgements

This research was supported by NSF CAREER award DRL-1350281.

References

Behrend, J. L., & Mohs, L. C. (2005/2006). From simple questions to powerful connections: A two- year conversation about negative numbers. *Teaching Children Mathematics*, *12*(5), 260–264.

Bishop, J. P. Lamb, L. L., Philipp, R. A., Schappelle, B. P., & Whitacre, I. (2011). First graders outwit a famous mathematician. *Teaching Children Mathematics, 17*(6), 350-358.

Bofferding, L. (2014). Understanding negative integers: Characterizing first-graders' mental models. *Journal for Research in Mathematics Education, 45*(2), 194-245.

Bofferding, L., Farmer, S., Aqazade, M., & Dickman, K. (2016). Leveraging contrasting cases: Integer addition with second graders. Manuscript under review.

Durkin, K., & Rittle-Johnson, B. (2012). The effectiveness of using incorrect examples to support learning about decimal magnitude. *Learning and Instruction, 22*(3), 206-2014.

Gentner, D. (2005). The development of relational category knowledge. In D. H. Rakison & L. Gershkoff-Stowe (Eds.), *Building object categories in developmental time* (pp. 245–275). Mahwah, NJ: Lawrence Erlbaum Associates.

Lange, K. E., Booth, J. L., & Newton, K. J. (2014). Learning Algebra from Worked Examples. *The Mathematics Teacher, 107*(7), 534-540.

National Governor's Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington DC: Author. Retrieved from http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Rittle-Johnson, B., & Star, J. R. (2007). Does comparing solution methods facilitate conceptual and procedural knowledge: An experimental study on learning to solve equations. *Journal of Educational Psychology, 99*(3), 561–574.

Rittle-Johnson, B., & Star, J. R. (2009). Compared with what? The effects of different comparisons on conceptual knowledge and procedural flexibility for equation solving. *Journal of Educational Psychology, 101*(3), 529–544.

Rittle-Johnson, B., & Star, J. R. (2011). The power of comparison in learning and instruction: Learning outcomes supported by different types of comparisons. In J. P. Mestre & B. H. Ross (Ed.), *Psychology of learning and motivation: Cognition in education* (Vol. 55). Elsevier.

Star, J. R., & Rittle-Johnson, B. (2009). It pays to compare: An experimental study on computational estimation. *Journal of Experimental Child Psychology, 102*, 408-426.

Tsang, J. M., Blair, K. P., Bofferding, L., & Schwartz, D. L. (2015). Learning to "see" less than nothing: Putting perceptual skills to work for learning numerical structure. *Cognition and Instruction*, 33(2), 154-197.

Vlassis, J. (2004). Making sense of the minus sign or becoming flexible in 'negativity'. *Learning and Instruction*, 14(5), 496-484.