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PK 12

Tiering Instruction for Seventh-Grade Students

A study based on a proportional reasoning unit shows how differentiating for students' ways of thinking can effectively meet diverse learning needs.

Amy J. Hackenberg, Robin Jones, and Rebecca Borowski

Are you being asked to differentiate instruction? Many school districts have such initiatives amidst calls to discontinue tracking (NCTM 2018). Yet teaching students with diverse ways of thinking in a single classroom must be done with great skill and care to address students' different learning needs (Mevarech and Kramarski 1997; Rubin 2008).

Differentiating instruction (DI) is a pedagogical approach to managing classroom diversity in which teachers proactively adapt curricula, teaching methods, and products of learning to address individual students' needs in an effort to maximize learning for all (Tomlinson 2005). DI is rooted in formative assessment, positions teachers and students together as learners, and involves providing choices and different pathways for students.

Although teachers can differentiate for many characteristics of students, we differentiate for students' diverse ways of thinking. Our definition of DI in mathematics classrooms is *proactively tailoring instruction to students' mathematical thinking while developing a cohesive classroom community* (Hackenberg, Creager, and Eker, under review).

In this article, we describe an example of DI involving middle school students from a five-year project funded by the National Science Foundation. In one phase of the project, a classroom teacher and the research team differentiated instruction for a class of 18 seventh-grade students during a 26-day unit on proportional reasoning. This class was for students deemed to be working on grade level, making this article a case of DI in a "typical" seventh-grade classroom. In addition, in our view of DI, listening to and interpreting students' thinking is central, making this article also a case of DI in which paying attention to students' mathematical thinking and developing practices to differentiate instruction are intertwined.

We present a portion of the unit, days 9 to 13, in which we began an investigation of speed and tiered instruction. In tiered instruction, teachers provide different groups of students with different problems that address the same big ideas (Pierce and Adams 2005) or, in our case, the same problem with different numbers. Tiering occurs after teachers have gotten to know students' thinking in a domain; they see a variety of thinking that will not be supported well with a one-size-fits-all approach; and they have ideas about problems that may support different groups. We tiered in our unit for these reasons.

We first explain how we thought about students' diverse ways of thinking. Then we show how we tiered instruction and how students worked on the problems, with accompanying video and audio clips. Finally, we discuss outcomes for students and teachers, and we provide some advice about DI.

We began an investigation of speed and tiered instruction . . . provid[ing] different groups of students with . . . the same problem with different numbers.

STUDENTS' DIVERSE WAYS OF THINKING: UNITS COORDINATION STAGES

Students come to middle school with three broad methods of organizing numbers and quantities into units (Steffe 2017). We refer to these ways of thinking as *stages* (Hackenberg, Norton, and Wright 2016) because transitioning between them requires significant learning that can take several years (Steffe 2017). The stages influence students' understanding of topics relevant to middle school learning goals, such as fractions (Hackenberg and Tillema 2009), integers (Ulrich 2012), and algebra (Hackenberg and Lee 2015). We use these stages to help us understand students' diverse forms of thinking.

Amy Hackenberg, ahackenb@iu.edu, is an associate professor of mathematics education at Indiana University. She studies relationships between students' rational number knowledge and algebraic reasoning, as well as how to differentiate mathematics instruction for diverse learners.

Robin Jones, robijone@iu.edu, is a doctoral candidate in mathematics education at Indiana University. She is interested in studying the influence of grades 4–7 curriculum on middle school students' understanding of proportionality.

Rebecca Borowski, borowsr@wwu.edu, is an assistant professor of mathematics education at Western Washington University. She studies students' quantitative reasoning, particularly on linear representations such as number lines.

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In this article, units are measurement units such as inches or nonstandard units (Ulrich 2015). Consider this problem:

> The classroom door measures eight chopstick lengths. There are seven paper clip lengths in a chopstick length, so how many paper clip lengths will measure the door's height?

Students at each stage tend to solve problems in different ways (see table 1). For example, students at stage 1 often reason with only one quantity at time rather than linking two together. In contrast, students at stages 2 and 3 often scale both quantities by the same number. Steffe (2017) estimates that at the start of sixth grade, 30 percent of students are at stage 1, 30 percent are at stage 2, and 40 percent are at stage 3. In our project, we investigated how these stages influenced students' ratio reasoning. Table 1 shows the results of what we found in our seventh-grade classroom.

DAYS 9 TO 13: THE SPEED INVESTIGATION

By day 9, we had assessed students' stages of units coordination and conducted formative assessment of students' ratio reasoning (see table 1). Because students at different stages were reasoning about ratios differently, we tiered instruction, conjecturing that different number choices could challenge students to make advances. For days 9 to 13, students worked in small groups that were relatively homogenous by stage. Although this class was relatively small, similar work can be done in larger classes, but there will be more small groups.

On day 9, we began the quantifying speed investigation, which we designed on the basis of a proportional reasoning unit by Lobato and colleagues (mathtalk.sdsu.edu). We used NewRace, a GeoGebra app designed by Janet Bowers. Students can enter a distance and time for each of two cars and run the race (see figure 1).

On days 9 and 10, students worked on tasks in which they were to make the red car go slower than the black car, given certain information. We invite readers to try these tasks now using the app:

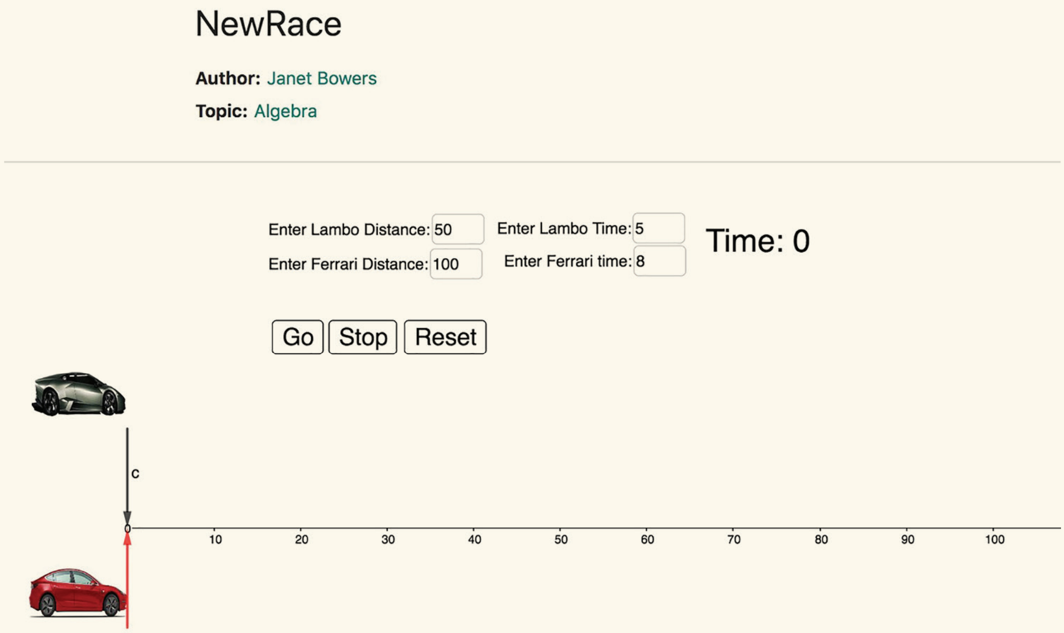
(1) The black car travels 15 miles in 6 minutes, and the red car travels 15 miles in ___ minutes. Find a time to make the red car go slower.
(2) Write a rule that would tell you how to choose the number of minutes to enter for the red car so that it goes slower than the black car when the two cars travel the same number of miles. Explain your rule.

Table 1 Units Coordination Stages

Stage	Typical solution for the chopstick/paper clip lengths problem	Typical view of the result of 56 paper clip lengths	Ratio reasoning	No. of students in class
1	Count on by ones past known skip-counting patterns for sevens	No multiplicative relationship between 1 paper clip length and the answer of 56	Often reason with only one quantity at a time	5
2	Use additive strategies to accumulate the 8 sevens; e.g., $7 + 7 = 14$; $14 + 7 = 14 + 6 + 1 = 20 + 1 = 21$, etc.	1 paper clip length × 56	Scale both quantities by whole numbers	9
3	Use multiplicative strategies; e.g., $8 \times 5 + 8 \times 2$	1 paper clip length × 56 and 7 paper clip lengths × 8	Scale both quantities by whole numbers and fractions	4

In our typical seventh-grade classroom, we investigated how at each stage the ways in which students tend to solve a problem can influence their reasoning about ratios.

Fig. 1



NewRace is a GeoGebra app that can be accessed at https://www.geogebra.org/m/vabtrttr.

(3) The black car travels 15 miles in 6 minutes, and the red car travels ___ miles in 6 minutes. Find a distance that makes the red car go slower.

(4) Write a rule that would tell you how to choose the number of miles to enter for the red car so that it goes slower than the black car when the two cars travel the same number of minutes. Explain your rule.

On days 11 through 13, students were given a distance and time for the black car, and they were to make the red car go the same speed using a different distance and time. They were then asked to justify their claims using pictures and explanations. We selected distances and times strategically for different thinkers (see table 2). For all students, we chose numbers for which taking thirds would produce the smallest whole number pair of values to create same speeds. We did so because we thought that halving might be a dominant strategy, and we wanted to see whether students would think to take thirds. We also attended to the nature of the unit ratio that would result, choosing numbers that would produce more basic to more complex unit ratios (see table 2). We invite readers to pick a pair of values and try the Same Speed task using the app.

STUDENTS' WORK

We now discuss the work of students at each stage on the Same Speed task. Students wrote in journals that had premade tables to record trials and results.

Table 2 Students' Stages and Numbers for the Same Speed Task

Stage	Task The black car goes—
1	18 mi. in 3 min.; unit ratio is a whole number (6 mi. per 1 min.)
2	15 mi. in 6 min.; unit ratio is a mixed number with 1/2 (2.5 mi. per 1 min.)
3	15 mi. in 9 min.; unit ratio is hard to work with as a decimal (5/3 mi. per 1 min.)

In our typical seventh-grade classroom, we investigated how at each stage the ways that students tend to solve a problem can influence their reasoning about ratios.

Students at each stage tend to solve problems in different ways.

Stage 1 Student: Emily

When Emily and her two group members tried to find a distance and time for the red car to go the same speed as the black car traveling 18 miles in 3 minutes, Emily suggested the following: 9 miles in 6 minutes, 18 miles in 6 minutes, and 18 miles in 2 minutes. She seemed to be halving or doubling either quantity but not operating on both together.

Then a group member suggested 36 miles in 6 minutes. Emily ran that race and was visibly excited when the cars kept pace with each other. She seemed suddenly subdued when the red car continued traveling after the black car stopped, but the group concluded that the cars had gone the same speed and that doubling each number "worked."

The first author of this article and co-teacher of the class, Amy Hackenberg (Ms. H), asked the group to draw a picture to justify why traveling 36 miles in 6 minutes was the same speed as traveling 18 miles in 3 minutes. No one initially had ideas. Ms. H asked if they could draw something to represent each trip. Emily's pictures evolved the most, and so we focus on her. Video 1 demonstrates Emily's progress in interaction with Ms. H. As the video shows, Emily's first picture (see figure 2a) demonstrated doubling only numerically with "× 2".

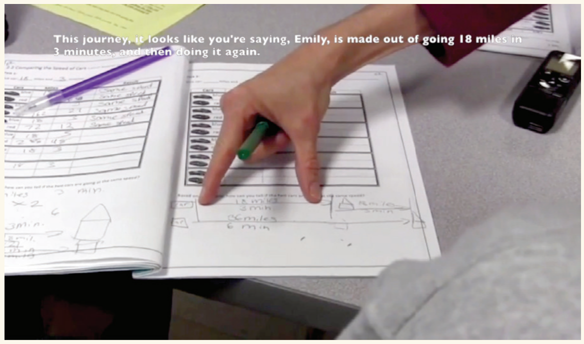
Then, when asked whether she could show the idea of doubling with lengths, she drew a picture with two lengths about the same size (see figure 2b).

When asked whether the journeys were the same size, Emily said no and extended the 36 mi.–6 min. segment. She did not make it exactly twice as long as the 18 mi.–3 min. segment, in part because she had reached the paper's edge (see figure 2c). She was about to draw a more exact picture when the period ended. Thus, Emily went from not knowing how to draw a picture to beginning to show how the 36 miles–6 minutes journey consisted of traveling the 18 miles–3 minutes journey twice (see figure 2c).

Stage 2 Students: Lisa and Sara

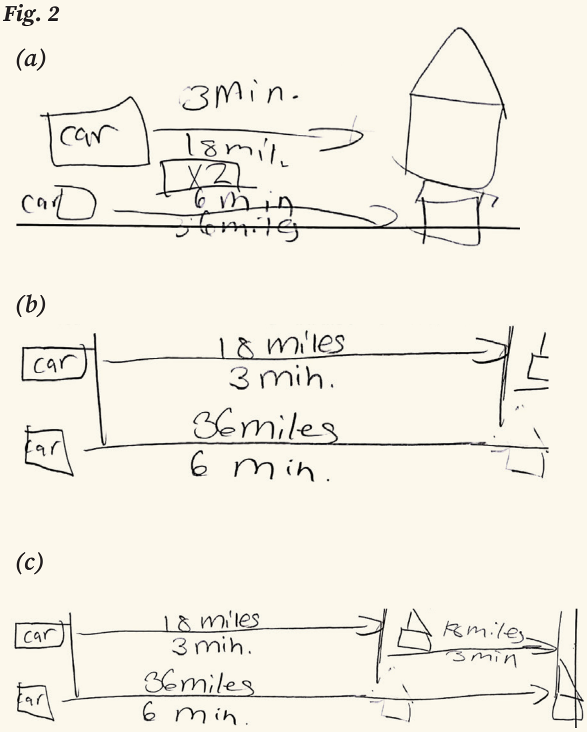
When Sara and Lisa tried to find a distance and time for the red car to go the same speed as the black car traveling 15 miles in 6 minutes, Lisa suggested 14 miles in 5 minutes, and then 15.1 miles in 6.1 minutes.

Video 1 Emily's Progress in Interacting with Ms. H



Watch the full video online.

Fig. 2



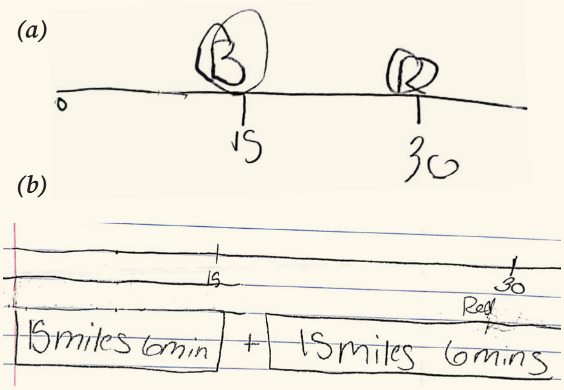
(a) In her first picture, Emily doubled only numerically with "× 2."
(b) In her second picture, the two journeys have the same lengths.
(c) The 36 mi.–6 min. journey is two 18 mi.–3 min. journeys.

The first author of this article . . . asked the group to draw a picture to justify why traveling 36 miles in 6 minutes was the same speed as traveling 18 miles in 3 minutes.

Fig. 3

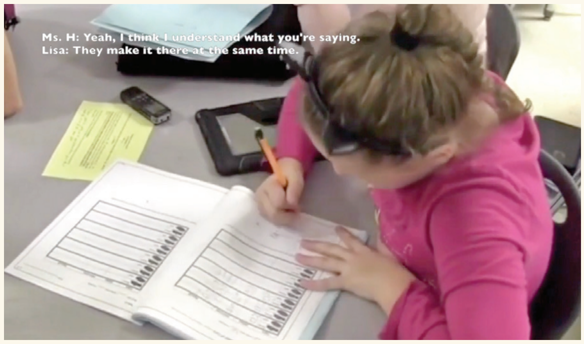
(a)

(b)



In Lisa's first picture, (a), the red car's distance is twice that of the black car. Her second picture, (b), shows two 15 mi.–6 min. journeys as the same as one 30 mi.–12 min. journey.

Video 2 Lisa and Sara's Justification



Watch the full video online.

After checking these on the app, both students said it was "impossible!" Ms. H asked if it was really not possible for two cars to travel the same speed but different distances and times. Sara said, "They probably could, but I can't figure it out." Then she added, "Unless you double it."

They ran a race in which the red car traveled 30 miles in 12 minutes and were excited that doubling both quantities produced the same speed. "I figured the system out!" proclaimed Sara. Lisa added that it might be possible to triple both quantities or use other multiples.

Like Emily and her group members, Lisa and Sara found it challenging to solve the task and justify their solution. In contrast to Emily, Lisa's first drawing (see figure 3a) showed the 30-mile distance as twice the length of the 15-mile distance. Video 2 demonstrates how Lisa and Sara developed their justification with Ms. H.

As video 2 shows, Ms. H noted that in Lisa's picture, it looked like the car traveling 30 miles went a journey of 15 miles in 6 minutes and then another journey of 15 miles in 6 minutes. Lisa agreed and drew another picture (see figure 3b) that showed the 30 miles–12 minutes journey as consisting of two "15 mi.–6 min." segments added together.

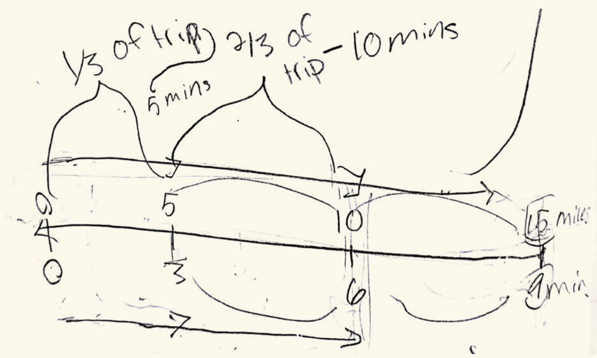
Lisa and Sara used this multiple-journey idea to explain solutions to this problem in a whole-class discussion the next day, as well as to solve other similar problems at the end of the unit. When Ms. H asked if they could find smaller distance-time pairs that would produce the same speed, they halved repeatedly and justified these solutions. So, Lisa and Sara went from thinking the task was impossible to creating and justifying multiple solutions.

Stage 3 Student: Joanna

When Joanna's group began discussing possible distances and times for the red car to go the same speed as the black car traveling 15 miles in 9 minutes, Joanna suggested 5 miles in 3 minutes. Mark, a group member, suggested 16 miles and 10 minutes. Joanna argued against this as audio 1 demonstrates (in the online supplemental audio file). Joanna's conclusion was that any numbers "where the miles would reduce to 5 and the minutes would reduce to 3" should work "because they're the same ratio to each other." She suggested 10 and 6 as another same-speed pair.

To justify her claim, Joanna drew distance and time segments partitioned into three equal parts of 5 miles and 3 minutes (see figure 4). Then she used her picture to justify same speeds. In the online supplemental audio 2 file, Joanna articulates the 5 mi.–3 min. segment to be one-third of the black car's trip. To Joanna, the 15 miles–9 minutes trip was a unit that could be partitioned into 5 mi.–3 min. segments, and she saw that any journey made from a multiple of one segment would have to be the same speed as the black car—a general way of thinking. She created this general way of thinking by determining the smallest whole number pair that would yield the same speed as the 15 miles–9 minutes journey.

Fig. 4



Joanna justified that multiples of 5 mi.-3 min. journeys produce the same speed as traveling 15 miles in 9 minutes.

CONCLUSION

Tiered instruction worked well in this part of our unit because students learned how to create and justify same speeds, but they did not all develop the same strategy. The different numbers helped us learn about students and DI in an intertwined way, which leads to our recommendations about DI.

Student Learning

All students found the Same Speed task challenging. For example, students at stage 1 or 2 (Emily, Lisa, and Sara) did not initially know how to solve it but then proposed doubling. However, they differed in their use of pictures to represent and justify their reasoning. Emily needed teacher support to show doubling quantitatively in her picture. In contrast, Lisa and Sara needed support to articulate the idea that a larger journey can be thought of as multiple smaller, same-speed journeys.

The nature of students' solutions also differed. For example, Lisa found a general solution by multiplying both given quantities by the same whole number, as well as by repeated halving. In contrast, Joanna found the smallest pair of whole numbers that would produce the same speed. Both solutions are general, but as teachers we might prefer Joanna's because it uses the structure of the numbers in a way that Lisa's does not, and it could lead nicely into work on unit ratios. However, Joanna's solution may be accessible primarily to stage 3 students—an issue for further research.

Teacher Learning

The different numbers also helped us learn as teachers. All students could have demonstrated their strategies with different numbers; these are not the only number choices we could have made. For example, the stage 1 and 2 students surprised us by not taking thirds; they could have used many distance-time pairs to accomplish their doubling solutions. However, their distance-time pairs brought to the surface a nice contrast between what we anticipated and what they did, allowing us to learn more about how to shape number choices for them in the future.

Joanna also could have shown her strategy with any pair of numbers that were multiples of three. However, working with the pair that produced the most complex unit ratio was important for her future learning: A next step is to think about how to take thirds again to produce 5/3 mile in 1 minute.

Our learning from this case leads to our recommendations about DI. In this kind of differentiation, teachers have an orientation of inquiring into students' thinking, making conjectures, and posing problems to explore the conjectures. We are reluctant to say that a particular number choice always works for students at a particular stage. We recommend that teachers get to know their students' thinking, trying different number choices, and observing the outcomes. One tool that teachers can use to get to know students' thinking is students' stages, and assessments are available

(Hackenberg, Norton, and Wright 2016).

In conclusion, our goal in DI is to provide each student with challenges that are sensible to them so they are working at the edges of their current reasoning. This instruction contrasts with a one-size-fits-all approach that overwhelms some students and underchallenges others. With this tiered lesson, we were able to better see the edges of students' thinking, pose appropriate challenges, and support students at different stages to develop their ideas. We view DI to be an important component of inclusive classrooms in which "equity is a priority" (Michael 2015, p. 82) because all students are seen as mathematical thinkers and receive what they need to learn.

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